Numerics of the ECMWF wave model

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INTRODUCTION

In this talk I would like to discuss how in the ECMWF wave model (which is a sibling of the **WAM** model) the adiabatic part of the energy balance equation is treated numerically.

But first I will present the basic evolution equation for the spectrum of ocean waves, and I will make plausible why the spectrum is advected by the group velocity.

The numerical treatment of the energy balance equation is then illustrated by means of the simple example of advection of the wave energy spectrum in one dimension. In the present operational setting a **first-order upwinding scheme** is used.

Arguments are presented why such a crude scheme is appropriate to model the advection of wave energy over large distances (of the order of $10,000$ km, while the typical wave length is of the order of $100-500$ m).

THE ENERGY BALANCE EQUATION and WAVE FORECASTING

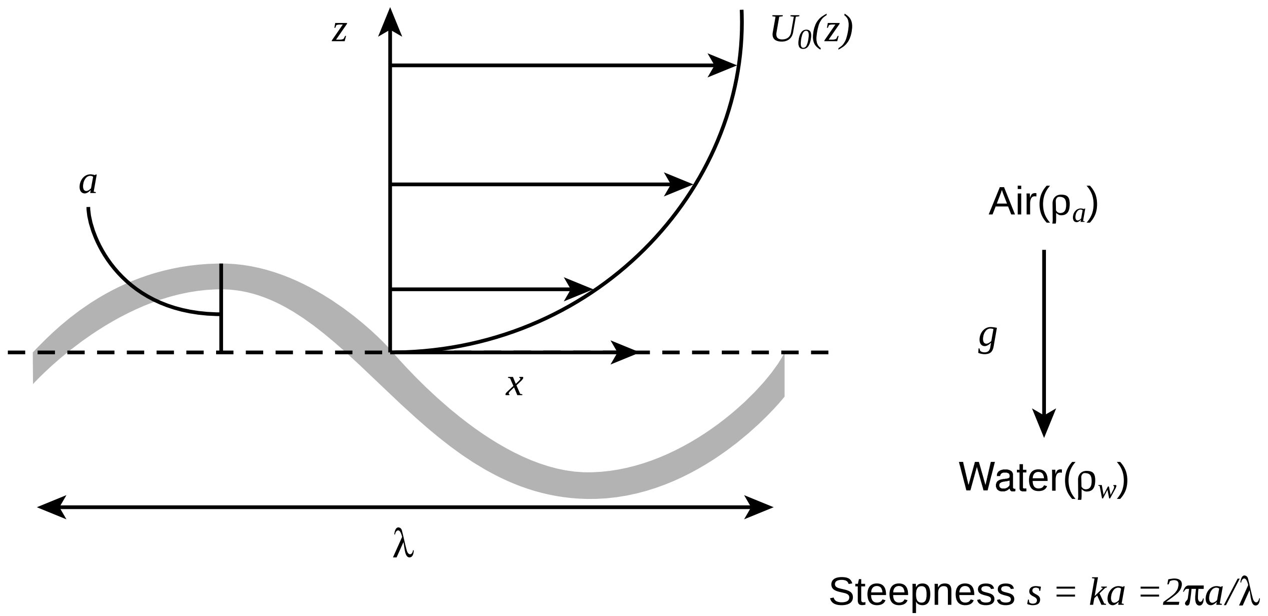


Figure 1: Schematic of the problem in two dimensions.

Consider the ideal case of surface gravity waves with vanishing air density, while the water density is constant. Furthermore, ignore effects of surface tension and viscosity.

Irrotational flow, $\nabla \times \vec{u} = 0$, gives velocity field in terms of a velocity potential ϕ,

$$\vec{u} = \nabla\phi$$

Because of the constant water density the water is divergence free, i.e. $\nabla.\vec{u} = 0$, hence inside the fluid the velocity potential satisfies Laplace's equation,

$$\nabla^2\phi = 0.$$

The main problem with water waves is, however, with the boundary conditions at the surface, described by $z = \eta(\vec{x}, t)$. First of all, the surface evolves in space and time. In the linear approximation one finds

$$\partial\eta/\partial t = w = \partial\phi/\partial z$$

The boundary condition at the surface then becomes the vanishing of the water pressure, where for potential flow the water pressure follows *Bernoulli's* law, $p = p_0 - \rho_w\,(\partial\phi/\partial t + gz)$. Hence,

$$\partial\phi/\partial t + g\eta = 0, \text{ at } z = 0.$$

Finally, assuming deep water, the velocity vanishes as z approaches $-\infty$.

The above problem can be solved exactly and for normal modes with wave number $k = 2\pi/\lambda$ and angular frequency $\omega(k)$,

$$\eta = a\exp(ikx - i\omega t),$$

the solution exists provided the angular frequency ω satisfies the dispersion relation

$$\omega(k) = \pm\sqrt{gk}.$$

Gravity waves have the important property that each wave number k has a different phase speed $c = \omega/k = \sqrt{g/k}$. Therefore, there is *Dispersion*. Furthermore, the longer the waves the faster they propagate. Finally, wave energy is propagating with the group velocity v_g,

$$v_g = \partial\omega/\partial k = \frac{1}{2}\sqrt{g/k},$$

hence, the group speed is exactly half the phase speed for deep water waves.

ENERGY BALANCE EQUATION

So far a single wave, but in nature there is a spectrum of waves. However, the phase of the waves is i) unknown, and ii) attempts to predict them gives chaotic behaviour. Therefore concentrate on the prediction of the ensemble average of the action density spectrum $N(\vec{k};\vec{x},t)$. Action plays the role of a number density and is defined in such a way that energy spectrum $F(\vec{k};\vec{x},t)$ is given as

$$F(\vec{k};\vec{x},t) = \omega(\vec{k}) \times N(\vec{k};\vec{x},t)$$

which is the usual rule in wave mechanics. From first principles one finds the following evolution equation

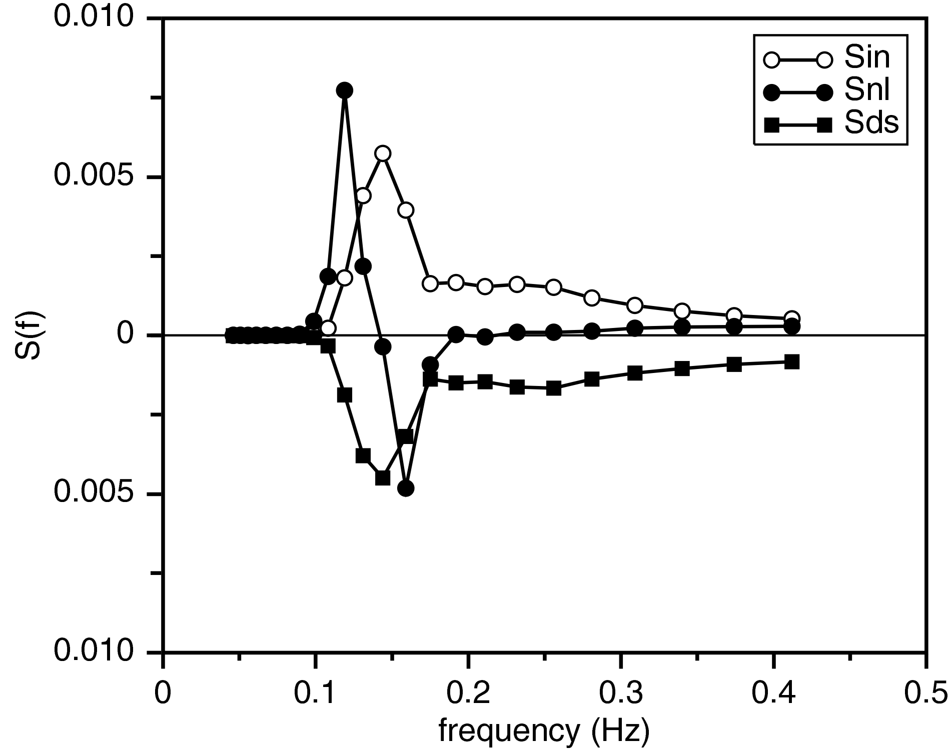
$$\boxed{\frac{\partial}{\partial t}N + \nabla_{\vec{x}} \cdot (\dot{\vec{x}}N) + \nabla_{\vec{k}} \cdot (\dot{\vec{k}}N) = S = S_{in} + S_{nl} + S_{ds},}$$

where $\dot{\vec{x}} = \partial\omega/\partial\vec{k}$, $\dot{\vec{k}} = -\partial\omega/\partial\vec{x}$, and the source functions S represent the physics of wind-wave generation, dissipation by wave breaking and nonlinear four-wave interactions. In this talk I will only discuss the adiabatic part, i.e. the L.H.S., of the energy balance equation.

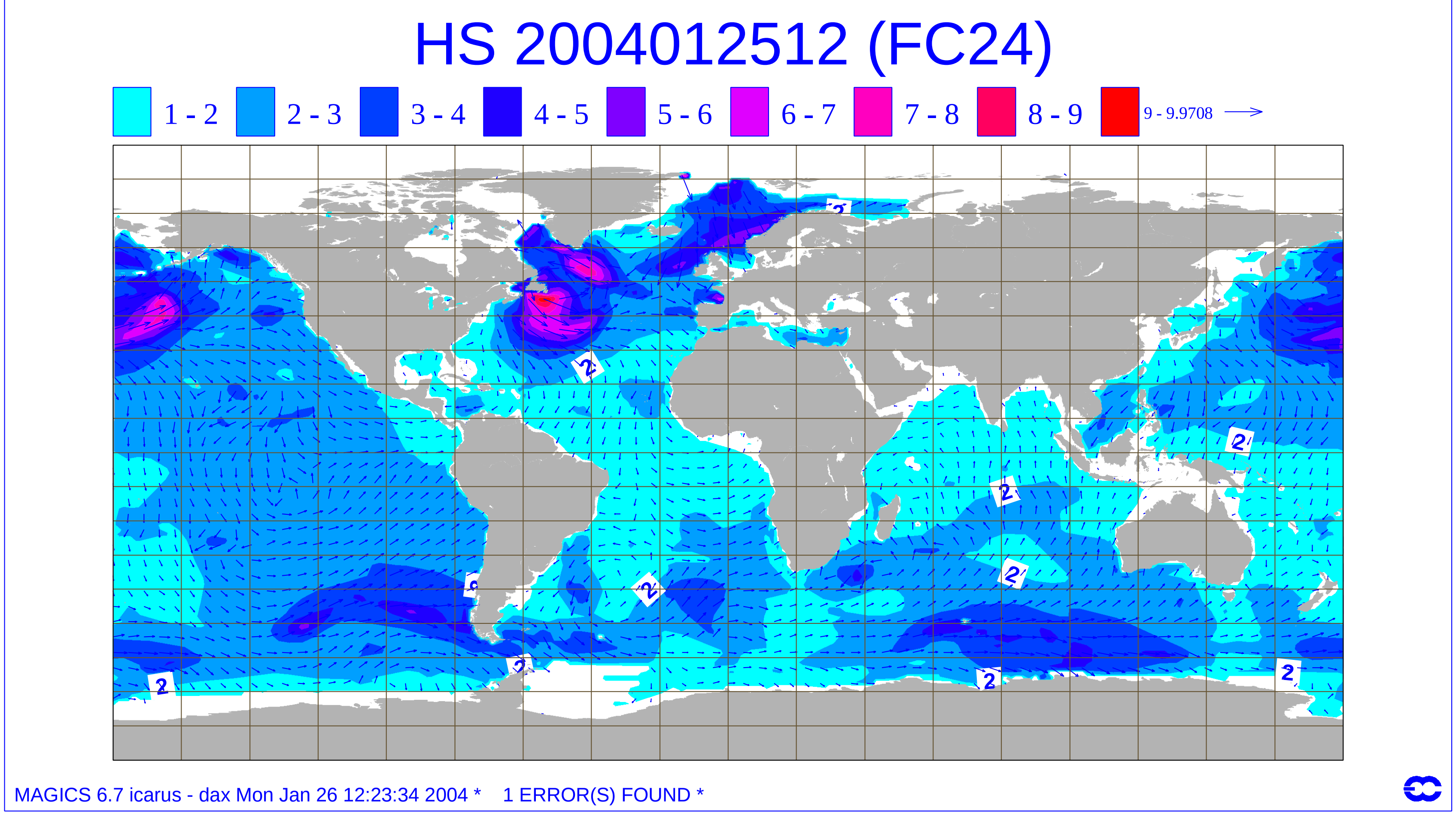
WAVE FORECASTING

The energy balance equation (including nonlinear transfer) is solved by modern wave prediction systems, where the forcing of the waves is provided by surface winds from weather prediction systems. For a large part ($\pm 80\%$) the quality of the wave forecast is determined by the accuracy of the surface wind field.

- An example of balance of source functions.
- Example of the forecast of wave height and wave spectrum.
- Verification of forecasts against the analysis.



The plot shows the energy balance of wind-generated ocean waves for a duration of 3 hrs, and a wind speed of 18 m/s.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Sunday 25 January 2004 12UTC ECMWF Forecast t+24 VT: Monday 26 January 2004 12UTC
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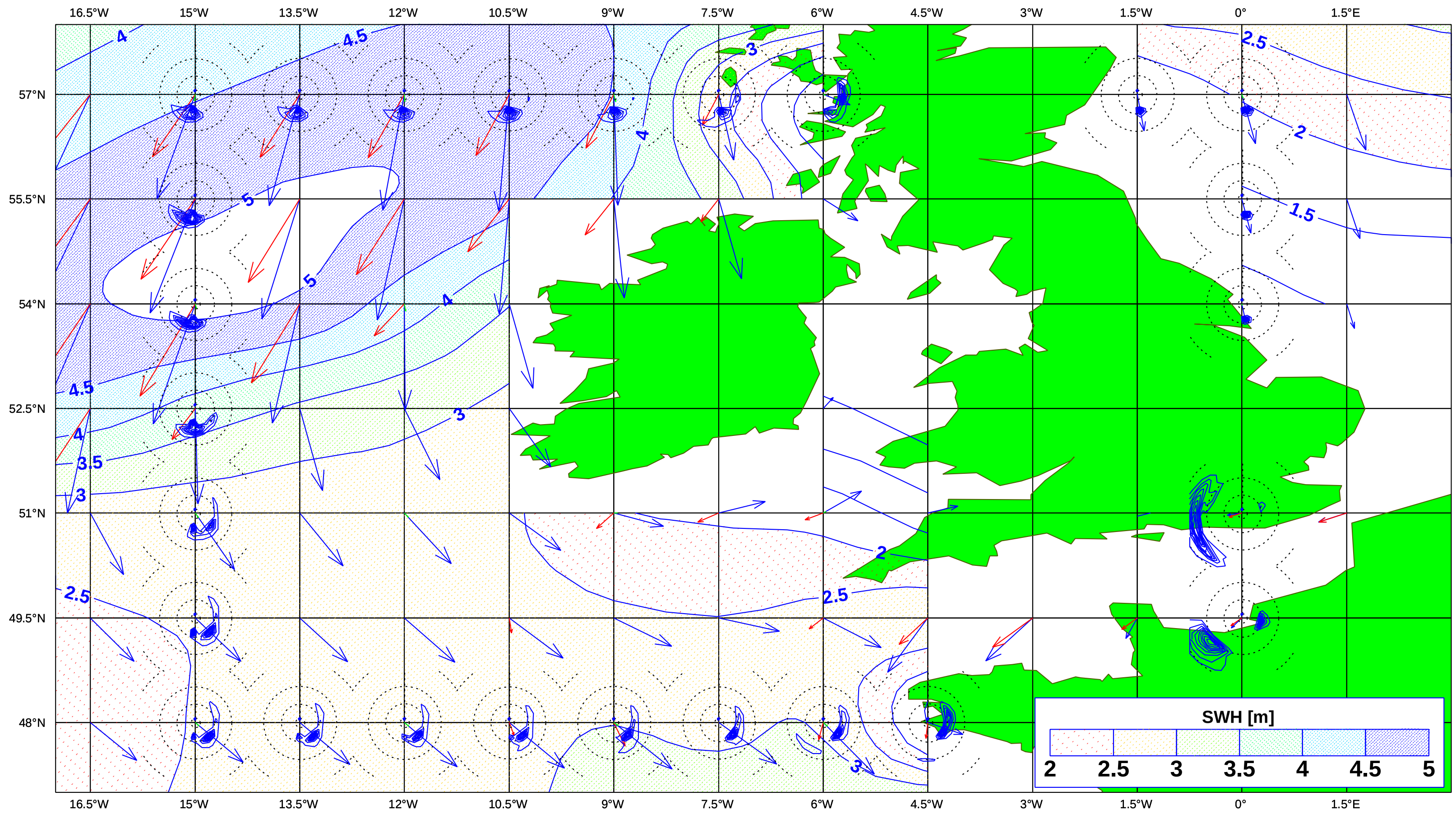

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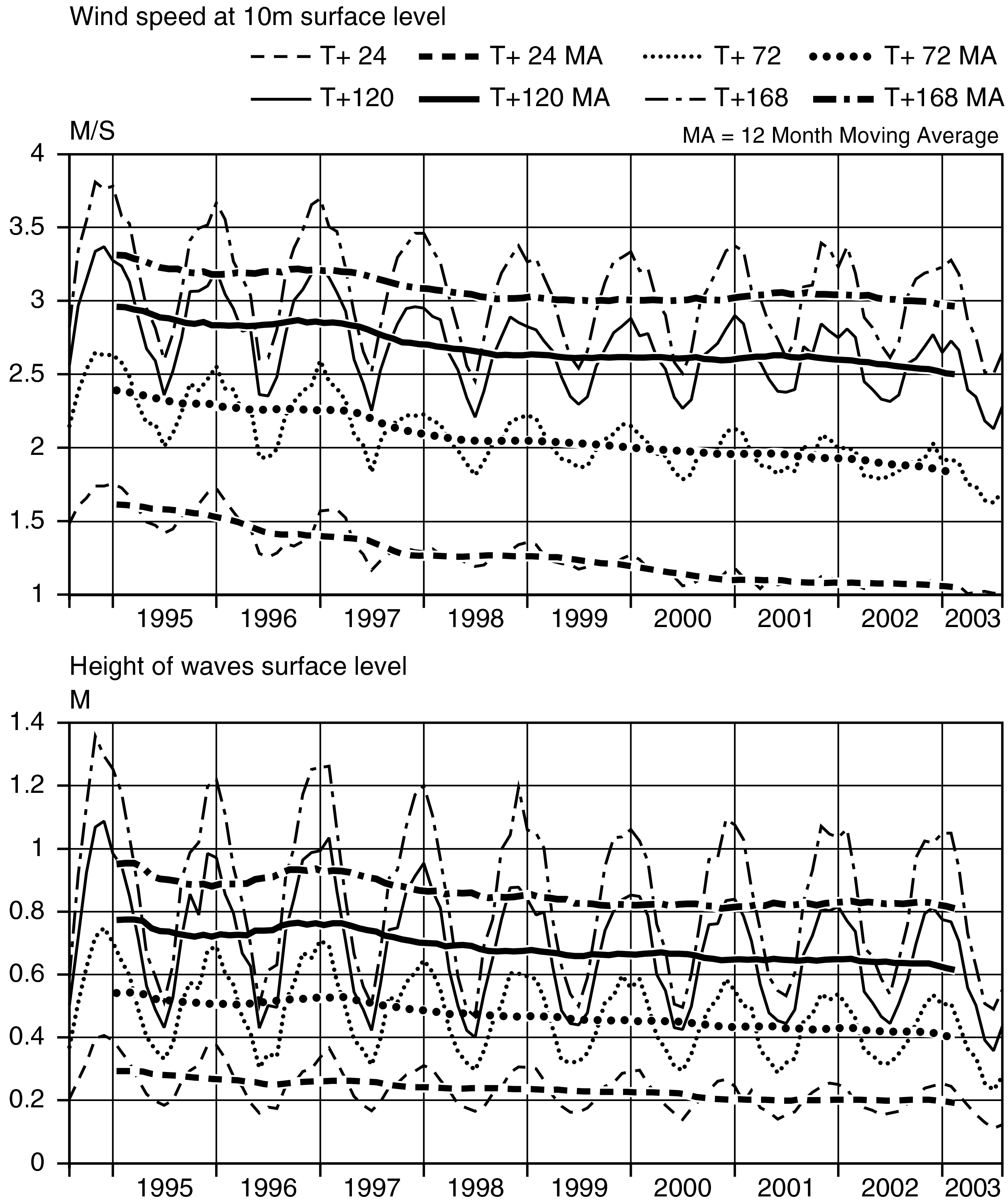


Figure 2: Forecast Verification against analysis for wind and waves in the Northern Hemisphere.

ADVECTION WITH THE GROUP SPEED: WHY?

From now on forget about the physics and refraction. Consider deep-water propagation in one dimension only, then evolution equation for the spectrum becomes

$$\boxed{\frac{\partial}{\partial t}F + v_g\frac{\partial}{\partial x}F = 0.}$$

Wave energy propagates with the group speed. This can be made plausible by studying the large time behaviour of the solution to the initial value problem:

$$\eta(x,t) = \int_{-\infty}^{\infty} d\kappa\; a(\kappa)e^{i(\kappa x-\omega(\kappa)t)},$$

where the wavenumber dependent amplitude $a(\kappa)$ follows from the Fourier transform of the initial condition $\eta(x,0)$. The content of this solution is hard to grasp, but the main features of dispersive waves follow by considering the asymptotic behaviour for large x and t; the interesting limit is $t \to \infty$ with x/t fixed.

Accordingly, the integral is written

$$\eta(x,t) = \int_{-\infty}^{\infty} d\kappa\ a(\kappa) e^{-i\chi t},$$

where

$$\chi(\kappa) = \omega(\kappa) - \kappa \frac{x}{t}.$$

For the present x/t is a fixed parameter. The integral may be studied by the method of stationary phase. Kelvin argued that for large time, the main contribution is from the neighbourhood of stationary points $\kappa = k$ such that

$$\chi'(\kappa) = \omega'(k) - \frac{x}{t} = 0.$$

Otherwise, the contributions to the integral oscillate rapidly and make little net contribution (**phase mixing** or **interference**). Expanding now $a(\kappa),\ \chi(\kappa)$ in the neighbourhood of the stationary point $\kappa = k$ one finds

$$a(\kappa) \simeq a(k),\ \chi(\kappa) \simeq \chi(k) + (\kappa - k)^2 \chi''(k).$$

With these approximations, the contribution becomes

$$a(k)\exp\{-i\chi(k)t\}\int_{-\infty}^{\infty} d\kappa \exp\left\{-\frac{i}{2}(\kappa-k)^2\chi''(k)t\right\}.$$

The remaining integral can be reduced to the real error integral

$$\int_{-\infty}^{\infty} dz e^{-\alpha z^2} = \left(\frac{\pi}{\alpha}\right)^{1/2}$$

and at the characteristic given by $x/t = \omega'(k) = v_g$ the solution becomes

$$\eta(x,t) \sim a(k)\sqrt{\frac{2\pi}{t|\omega''(k)|}}\exp\left\{ikx - i\omega(k)t - \frac{\pi i}{4}sign(\omega''(k))\right\}$$

Obviously, this approach only works for dispersive waves with $\omega''(k) \neq 0$.

In words, for large times the main contribution to the signal is found along the characteristic determined by the group velocity. This makes plausible that wave energy propagates along characteristics determined by the group speed, and justifies the use of the energy balance equation. However, it should be clear that this description is only valid for large times, which in practice means on timescales of the order of a number of wave periods.

NUMERICAL ADVECTION SCHEME

A number of alternative propagation schemes have been tested by different groups in the past decade. Examples are first-order upwinding schemes, a second order leap frog scheme, semi-Lagrangian schemes, third-order schemes, etc. However, none of the schemes give satisfactory results unless special measures are taken. In fact, a propagation scheme with vanishingly small errors would give poor results for sufficiently large propagation times since it would not account for the dispersion associated with the finite resolution of the wave spectrum in frequency and direction (the so-called Garden- Sprinkler effect).

In order to explain this, study the evolution of one spectral bin having a height $F_n(x,t)$ and a width Δf_n. The total wave energy E is therefore given by

$$E = \sum_{n=0}^{n=N} F_n \Delta f_n.$$

The energy in bin n propagates with the group velocity $v_{g,n}$, or

$$\frac{\partial F_n}{\partial t} + v_{g,n}\frac{\partial F_n}{\partial x} = 0,$$

For the initial condition

$$F_n(x,0) = f_n(x)$$

it is straightforward to solve for the evolution of the wave spectrum. The solution becomes

$$F_n(x,t) = f_n(x - v_{g,n}t)$$

hence the waves with group speed $v_{g,n}$ propagate over the surface with a spatial distribution that does not change its shape. Since this is a linear problem, the solution for an arbitrary number of spectral bins is obtained by summation of the solution for different group velocity. Consider the solution for two neighbouring frequency bins and suppose that the two bins have equal spatial distribution of the box type with width Δx, where Δx is the spatial resolution.

Clearly, after a finite time τ_s there is a separation of the two pulses which is determined by the difference in group velocity and the spatial width. Assuming that the frequency increment Δf_n is small, one may use a Taylor expansion of the difference in group velocity and the separation time τ_s becomes

$$\tau_s = \frac{\Delta x}{v_{g,n}} \frac{f}{\Delta f_n}$$

For larger time droplets are formed on the surface, hence the name Garden Sprinkler effect. In a similar vein, it can be shown that a finite directional resolution $\Delta\theta$ will give rise to the Garden Sprinkler effect as well. Finite directional resolution gives in practice a much shorter separation time since

$$\tau_s = \frac{\Delta x}{v_{g,n}} \frac{1}{\Delta\theta} \tag{1}$$

and $1/\Delta\theta < f/\Delta f_n$. It is emphasized that for a continuous spectrum the Garden Sprinkler effect will not occur and therefore in a discrete model measures have to be taken to avoid it.

The compromise

The Garden Sprinkler effect can be avoided by realizing that inside one bin the group speed varies with frequency. Working out the consequences one finds that in good approximation the the energy in a bin satisfies an advection diffusion equation of the type

$$\frac{\partial F_n}{\partial t} + v_{g,n}\frac{\partial F_n}{\partial x} = D\frac{\partial^2 F_n}{\partial x^2},$$

where the diffusion coefficient D is proportional to the age of the wave packets. This gives, as expected from dispersion, a linear spreading rate. However, it is an expensive solution because the age of each spectral bin is required.

The compromise now is to use schemes with a *constant* diffusion coefficient. This explains why a first-order upwinding scheme is so succesful, despite the fact that it is only first order accurate and is highly diffusive having a numerical diffusion coefficient $D \simeq (\Delta x)^2/\Delta t$ (with Δt the time step).

However, it is of course preferable to have some control over the amount of horizontal diffusion. Therefore, one has studied the benefits of higher-order schemes (which are, of course more accurate than the first-order schemes), combined with a smoothing of the results. In practice, the smoothing negates the advantage of the accuracy of the higher-order scheme.

Numerical Experiments

These findings are illustrated by means of the following numerical experiments, performed by Saleh Abdalla (Results for wave height $H_S = 4\sqrt{E}$):

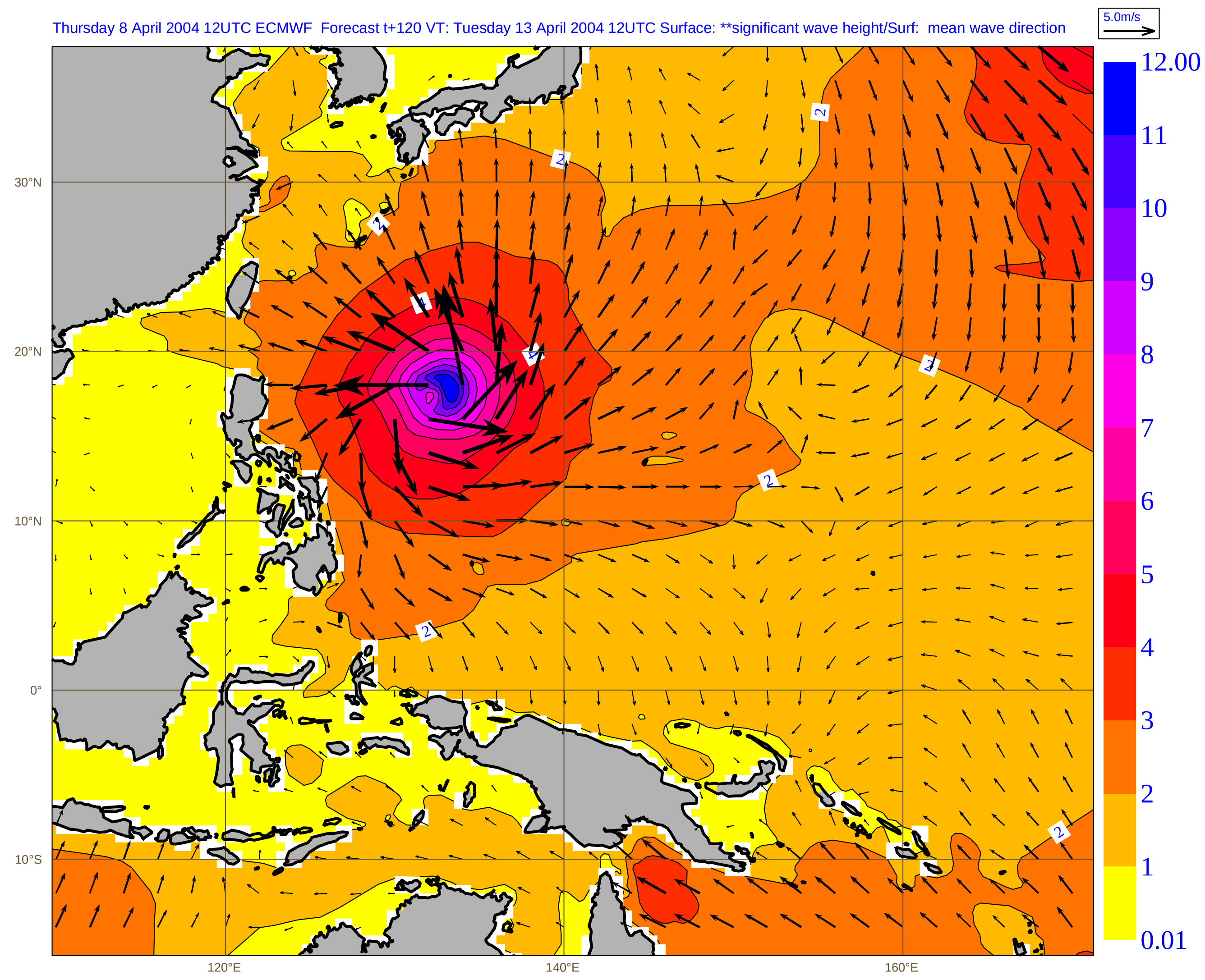
- one dimensional propagation test with Cubic-Interpolation Scheme (CIP). $\Delta f = 0.01 Hz$. This accurate scheme shows the Garden Sprinkler effect.
- CIP scheme with higher frequency resolution ($\Delta f = 0.001 Hz$). Removes Garden Sprinkler effect in practice. This is not feasible operationally.
- CIP scheme with a spatial smoother:

  $$F_n(x) = (1 - 2\alpha) F_n(x) + \alpha \left\{ F_n(x + \Delta x) + F_n(x - \Delta x) \right\}$$

  where $\alpha = 0.2 v_{g,n} \Delta t / \Delta x$. Removes Garden-Sprinkler effect.
- first-order upwinding scheme. No Garden-Sprinkler effect.

Therefore, the first-order upwinding scheme seems to perform reasonably well, and is used for many years now in the operational ECMWF wave model. It even performs well in extreme conditions such as hurricanes (cf. example).

However, first- and third-order upwinding schemes have the drawback that the small scales are heavily damped resulting in too low variability in the wave height field.


Thursday 8 April 2004 12UTC ECMWF Forecast t+120 VT: Tuesday 13 April 2004 12UTC Surface: **significant wave height/Surf: mean wave direction
5.0m/s
12.00
11
10
9
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6
5
4
3
2
1
0.01
30°N
20°N
10°N
0°
10°S
120°E
140°E
160°E

SMOOTHING and HIGH-FREQUENCY VARIABILITY

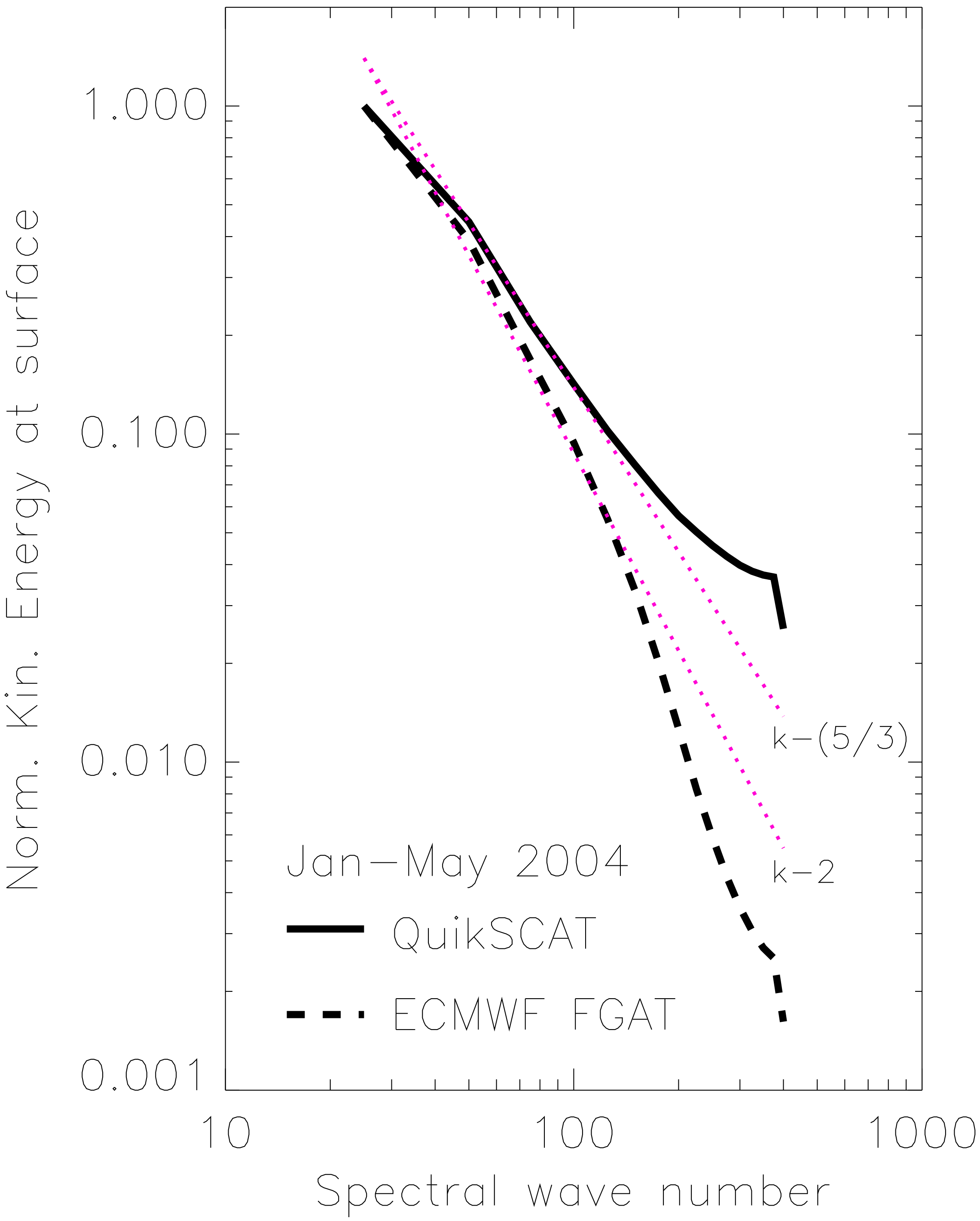
Interpolation (either linear or cubic) has the drawback that high-frequency variability is highly damped. For the shortest scales typical damping time scales τ_d are

$$\tau_d \simeq 2 \times \Delta t,$$

hence, damping by smoothing is for the small scales an important process.

This is one of the reasons why there is little variability in modelled wave height compared to observed wave height. Illustrated by observed (Altimeter) and modelled wavenumber spectrum of significant wave height.

Another reason is the too small variability in the *atmospheric* wind field. This follows, e.g., from a comparison of modelled and observed (by the scatterometer) wavenumber spectrum of the surface wind field.

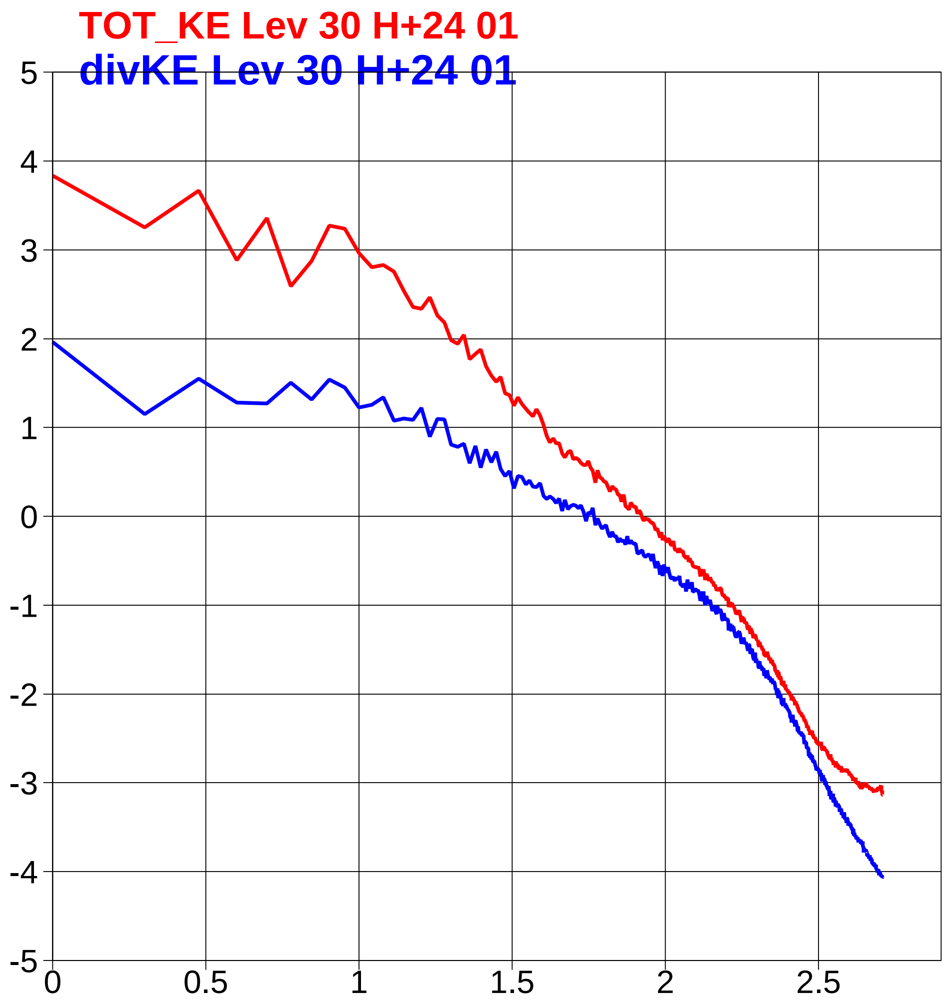


Norm. Kin. Energy at surface
1.000
0.100
0.010
0.001
10
100
1000
Spectral wave number
k−(5/3)
k−2
Jan−May 2004
QuikSCAT
ECMWF FGAT

As far as I can see the underestimation of modelled atmospheric activity in the short scales is most likely caused by the smoothing action of the interpolation in the semi-Lagrangian scheme.

Note that problems near the surface are relatively minor. Near the tropopause, the high-wavenumber part of the ECMWF modelled variability is considerably less than as found by aircraft observations from Nastrom and Gage (1985).

This underestimation of variability is a serious problem, if one, at least, is interested in a proper representation of the EPS spread (Shutts, 2004), of the interaction of ocean and atmosphere or in a proper representation of ocean waves.



Kinetic energy as function of total wave number on a logarithmic scale. The contribution by divergent flow is shown as well.

Aircraft data from Nastrom and Gage, 1985

Concluding Remarks

- I have demonstrated that an accurate propagation scheme not always leads to the desired result. For large propagation distances, the dispersion associated with the finite resolution of the wave spectrum in frequency and direction needs to be accounted for (the so-called Garden-Sprinkler effect).
- The Garden-Sprinkler effect can be avoided by introducing a spatial smoothing or by introducing (explicitly or implicitly) horizontal diffusion. This solution has, however, the drawback that the high-frequency variability of the wave field is reduced.
- Similarly, interpolation such as done in semi-Lagrangian schemes will results in large damping of the small scales. The conjecture is that this is the main reason that a $k^{-5/3}$ law is not found in the ECMWF atmospheric model.