NONLINEAR MOUNTAIN WAVES AND WAVE-MEAN FLOW INTERACTION: ELEMENTS OF A DRAG PARAMETERIZATION

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Summary: Several recent lines of argument and evidence have converged which point anew to the important role played by the interaction between internal gravity waves and the synoptic scale flow. The main source of such wave activity in the earth's atmosphere is that associated with stratified flow over mountainous terrain. Detailed analyses of the nature of the waves generated through this mechanism, and of the way in which they interact with the mean flow, demonstrate that conventional schemes for parameterization of the drag communicated by the waves to the mean flow are at best questionable. The work which has led to this conclusion is reviewed here and suggestions made based upon it which should eventually lead to the design of a new and more effective wave drag parameterization for use in general circulation models (GCM's).

1. INTRODUCTION

It has been generally well understood since the work of Sawyer (1959) that the force exerted on the earth when a field of internal waves is set up by flow over mountainous terrain may be comparable to, or even exceed, the direct frictional force over areas where significant small scale topography exists. It is a consequence of Newtons' second law that this force which the air exerts on the earth is not directly communicated to the mean flow of air at the surface; rather, the wave field generated by the topography transports counter-flow momentum in the vertical. It is now known

*John Simon Guggenheim Foundation Fellow that this momentum flux is actually delivered to the mean flow only under rather special (if inevitable) circumstances which occur when the waves "break". The issues of how and where the mean flow is affected in this circumstance will have to be settled before we will be in any position to accurately parameterize the influence of internal wave drag in a general circulation model.

Sawyer's original linear estimates of internal wave drag "potential" were confirmed by Blumen (1965) as being in the range 1-10 dyne cm^{-2} in conjunction with stationary waves having horizontal wavelengths in the range 10-100 km. Bretherton (1969) applied linear theory to estimate the drag delivered to the mean flow for an actual combination of three dimensional flow and topography at a specific geographic location and also obtained estimates similar to those of Sawyer. It is crucial to the appreciation of all that will follow in this review to understand that linear theory predicts that the only occasion in which the inviscid wavefield will give up its momentum to the mean flow is through the interaction which occurs at a critical level, where by definition the horizontal phase speed of the wave equals the speed of the mean flow (or in circumstances of strong wave transience which we will ignore). As had been demonstrated by Booker and Bretherton (1967), the nature of this linear interaction is governed entirely by the value of the gradient Richardson number Ri_c at the critical level itself ($Ri_c = N^2/(dU/dz)^2$ where N is the Brunt-Vaisala frequency and U the horizontal speed of the mean flow), which is such that the incident momentum flux is entirely absorbed to the extent that $Ri_c >> 0.25$. This is the concept from linear theory which Bretherton (1969) employed to estimate where and how much wave drag would be communicated to the mean flow in the case study which he described.

Recent analyses of nonlinear mountain waves (Clark and Peltier 1977, Peltier and Clark 1979, 1980, 1983, Clark and Peltier 1984) have very clearly established that Bretherton's linear analysis of the wave drag parameterization problem is severely limited in its applicability. These new analyses, which will be reviewed below, have demonstrated two important points which had not previously been understood. Firstly, they have established that the actual interaction between a wave and the mean flow at a critical level where $Ri_c >> 0.25$ is such that the wave is strongly reflected rather than being absorbed, a consequence of the nonlinear redistribution of vorticity which takes place in the critical layer itself. The second point which these recent analyses have established is that the only occasion in which strong wave - mean flow interaction occurs is when the wave "breaks" in the sense that streamlines are caused to locally overturn and thus local vertical temperature gradients are caused to exceed the dry adiabatic. The idea that breaking internal waves deposit momentum into the mean flow at the breaking level is an idea that was first introduced in the GFD literature in the context of efforts to understand dissipation processes in the upper atmosphere (Hodges 1969, Hines 1970). It has also been perceived more recently as being a crucial source of dissipation in the middle atmosphere, in which context Lindzen (1981) and Holton (1982) have revived Hodges original idea of "wave saturation" to develop a parameterization scheme on the basis of which the turbulent dissipation effected by breaking could be estimated if the incident spectrum of internal waves was known. Peltier et al. (1985, 1986) have shown that the phenomenon of sudden stratospheric warming (SSW) is extremely sensitive to the magnitude and spatial distribution of the dissipation which is assumed to act in the middle atmosphere and suggested that SSW might therefore be employed as a vehicle for testing possible schemes for the parameterization of wave drag.

One of the more important outcomes of the above cited new analyses of nonlinear mountain waves has been the demonstration that the basic assumption of the Lindzen-Holton (Hodges) parameterization scheme is most probably wrong. As we will see in what follows, although the saturation hypothesis does turn out to be correct, the momentum flux in excess of that required to maintain saturation is apparently not deposited into the mean flow where the wave breaks. What actually occurs is that the excess wave activity is reflected rather than being absorbed just as has been shown to be the case for the critical level interaction. It is upon the proper representation of this dynamical process within general circulation models, and in particular upon the understanding of its implications concerning wave-mean flow interaction, that the construction of a rational wave drag parameterization scheme must depend.

In fact the analyses of the evolution of topographically forced internal waves which we have performed were not directly motivated by the wave drag parameterization problem. Rather this work had its origins in a desire to understand a very specific atmospheric phenomenon, namely the occurrence of strong downslope windstorms which are variously known in different parts of the world as the Chinook (Canada), the Foehn (Switzerland), and the Bora (Yugoslavia). When we began this work the prevalent theory of these phenomena was that they were explicable in terms of a simple linear hydrostatic model (Klemp and Lilly 1975) which held that a strong downslope windstorm occurred whenever the atmospheric mean state was such that the phase shift of the forced internal wave across the troposphere was some integer multiple of $\Pi/2$, i.e. the height of the tropopause was some integer multiple of half vertical wavelengths. This theory was more fully elaborated in a number of later articles (Klemp and Lilly 1978; Lilly and Klemp 1979; etc.) in which the well observed Colorado front range windstorm of 11 January 1972 (Lilly and Zipser 1972) was employed to serve as a test of the theory's predictions. Our analysis of the same windstorm (Peltier and Clark 1979) led us, however, to an explanation of this dynamical event which was completely different from that which had been proposed by Klemp and

Lilly. This analysis established that in fact the process which was crucial to the ability of a numerical model to rather precisely reproduce the observations of the January 11, 1972 windstorm event was the occurrence of wave breaking. In that paper we described two model integrations which differed from one another only in that, for one, the stratospheric wind speed was increased above the observed speed by an amount just sufficient to prevent wave breaking. These models, for which the hydrostatic phase shift of the wave across the troposphere was identical, evolved in completely different ways such that only the model in which the wave actually broke in the lower stratosphere was able to reproduce the observations. Although this conflicting analysis initially led to a good deal of heated disputation in the open literature (Lilly and Klemp 1980; Peltier and Clark 1980) the Peltier and Clark analysis has since been reproduced by others and thus shown to provide the only viable explanation of the downslope windstorm phenomenon (Hoinka 1985; Durran 1986). Such debate as continues has become focussed on the interpretation of the processes which occur subsequent to wave breaking which are responsible for transforming the flow into one characterized by intense downslope flow in the lee of the topography (e.g. Durran 1986).

In Peltier and Clark (1979) an hypothesis was presented as to the physical process which was responsible for effecting this transformation. This hypothesis, which was suggested to the senior author of this paper on the basis of work on the stability of stratified parallel flows above the ground by Davis and Peltier (1976, 1977, 1979), was made more explicit in Peltier and Clark (1983). The basic idea underlying this hypothesis was that when a mountain wave field was forced to exceed critical steepness somewhere (to "break"), the flow thereafter became susceptible to an instability which consisted of a horizontally localized mode, resonant in the cavity between the level of breaking and the surface, which would have positive growth rates only when the basic state wave was supercritical in amplitude. Because this mode turns on just as streamlines overturn, all of the forcing in excess of that required to support the mode would project onto this secondary instability rather than onto the freely propagating wave. The observed linear growth of the wave in the cavity which was revealed by the numerical integrations could then be understood in terms of a model consisting of a simple harmonic oscillator forced at its resonant frequency (Peltier and Clark 1983). This hypothesis also provided an immediate explanation of the saturation phenomenon. The validity of this hypothesis has recently been more fully established through a series of detailed analyses of the stability of the nonlinear wave field (Laprise and Peltier 1986), upon which we will not have space to provide detailed comment here but to which we direct the interested reader for further discussion. In the following sections of this document we will review the basic results which led to the construction of this theory and will comment upon its implications for the problem of designing a parameterization scheme for the incorporation of mountain wave drag in models of the atmospheric general circulation.

2. LINEAR THEORY FOR MODELS WITH CONSTANT N AND U

The fundamental results in the linear theory of mountain waves were established in the papers by Quency (1941) and Lyra (1943). For a fully compressible flow characterized by constant N and U upstream of two dimensional topography $z_s(x)$ (where ξ is by definition such that $w = U\partial\xi/\partial x$ with w the perturbation vertical velocity) is:

$$\xi(x,z) = \Pi^{-1}\exp(z/H) \int_c \exp i[kx + (k_G^2-k^2)^{\frac{1}{2}} z^1] \, z_s(k) \, dk \qquad (1)$$

where

$$k_G^2 = \frac{(N^2/U^2) - \omega_a^2/C^2)}{[1 - U^2/C^2]}$$

$z^1 = [1 - U^2/C^2]^{\frac{1}{2}} z$

And c is the contour $0 \le k \le \infty$ along the real axis in the complex k-plane, $\omega_a = g/2C$ is the acoustic cut-off frequency, and C the adiabatic sound speed. In the limit of low Mach number $M = U/C << 1$, $k_G^2 \Rightarrow N^2/U^2$, $z^1 \Rightarrow z$ and (1) can be reduced to a form involving separate real quadratures in the upstream and downstream regions. An example of such a solution for symmetric topography is illustrated in Figure 1 which shows, in plates (a) and (b) respectively, contours of constant $\xi(x,z)$ and $w(x,z)$. Figure 2 shows the stream function field for this linear steady state flow with the height of the topography fixed so that the wave just exceeds critical steepness.

The height above the ground at which the streamlines first overturn may be quite accurately estimated using linear theory and most easily on the basis of the assumption that the wavefield may be assumed hydrostatic. In the hydrostatic, long wave limit $k_G^2 >> k^2$, and (1) may be analytically evaluated for symmetric topography $z_s(x) = a^2h/(x^2 + a^2)$ using Hilbert transforms to give (Miles and Huppert 1967):

$$\xi^L(x,z) = \frac{a^2h}{(x^2+a^2)} e^{z/2H} [\cos k_G z - \frac{x}{a} \sin k_G z] \qquad (2)$$

Now the condition for overturning of the streamlines is clearly $\partial\xi/\partial z > 1$, a condition which follows from the conservation of potential temperature $d\theta/dt = 0$ which, expanding $\theta = \theta_o + \theta'$, reduces in the steady state to the equation $\partial\theta'/\partial x + (\partial\xi/\partial x)(d\theta_o/dz) = 0$. On the basis of the assumption of no upstream influence this may be integrated directly to give $\theta' = -\xi x (d\theta_o/dz)$. Thus the stability of the basic state will be entirely offset by the wave where $(-\partial\theta'\partial z)_{max} > (d\theta_o/dz)$ or where $(\partial\xi/\partial z)_{max} > 1$. From (2) we see by inspection that extrema of $\partial\xi^L/\partial z$ are all found immediately overhead of the mountain (x=0) and the first steepening level z_c is where $k_G \times z_c = 3\Pi/2$. The wave induced critical region is therefore located at a height

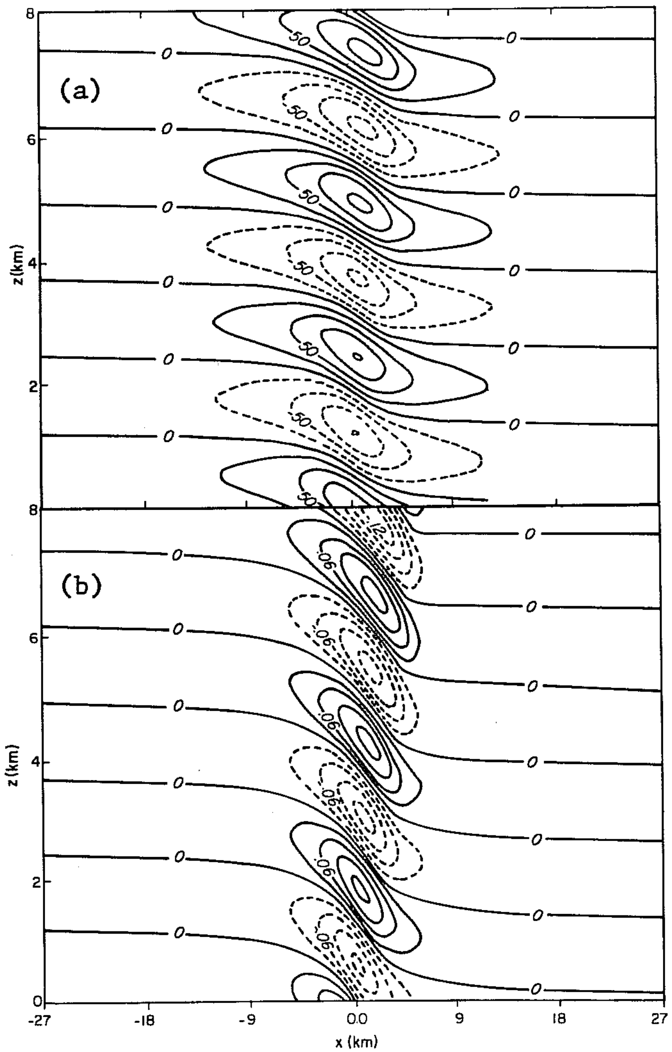

Figure 1a: Contours of constant (steady state) free stream deflection $\xi(x,z)$ based upon linear inviscid theory. The parameters of the model are $2\Pi/N = 10.2$ min, $U = 4$ m s^{-1}, $a = 3$ km, $h = 100$ m. The contours are labelled in m with solid lines positive and dashed negative.

Figure 1b: Contours of constant (steady state) vertical velocity w derived from $w = U\partial\xi/\partial x$ and the data of part (a).

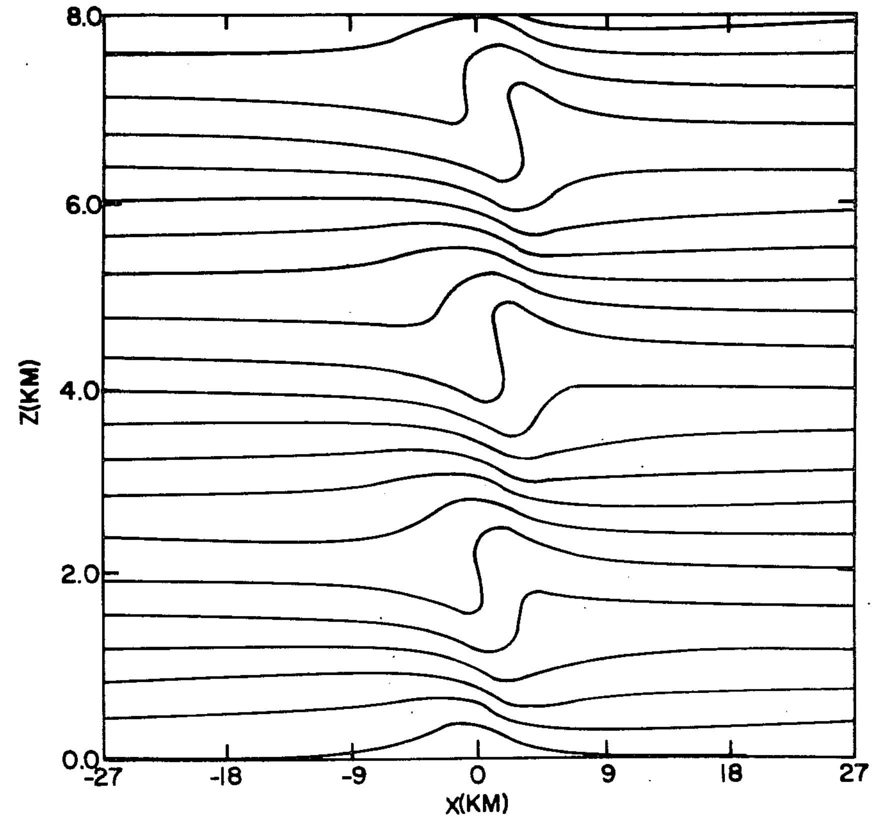

Figure 2: Linear steady state (inviscid) streamline pattern for the same model employed for Figure 1 except h = 500 m.

$z_c = 3\lambda_z/4$ where $\lambda_z = 2\Pi/k_G$ is the vertical hydrostatic wavelength of the wave.

Although the prediction from linear theory of the height at which wave overturning first occurs is rather accurate, even for flows with variable N and U in which case the WKBJ generalization

$$\int_0^{z_c} k_G(z')\, dz' = 3\Pi/2$$

may be employed, the linear prediction of the height of the topography necessary to induce this condition may be extremely inaccurate. For example, the results in Figure 2 were obtained from a linear calculation with h = 500 m, but inspection of the apparent topography based upon the relief of the streamline which enters the domain at z=0 upstream is considerably less than this. According to this linear theory the height of the topography necessary to induce the critical condition corresponds to a height h = 400 m or to a Froude number Fr = h/U/N) ≃ 1.00 where the Froude number is clearly the ratio of the height of the topography to the vertical hydrostatic wavelength of the forced internal waves. As we will see in the next section, however, for symmetric topography $z_s = a^2h/(x^2 + a^2)$ the critical Froude number $Fr_c = 0.85$ so that the error in the linear prediction of the critical topographic height is greater than 20%.

3. LONG'S MODEL: EXACT NONLINEAR SOLUTIONS WITH CONSTANT N AND U

Long (1953) was the first to realize that under certain circumstances, which essentially reduce to constant N and U, it is possible to obtain exact nonlinear steady state solutions to the mountain wave problem. He showed that the solution to the full non-linear Boussinesq problem, in terms of the free stream deflection ξ, was provided by the solution to the equation:

$$\nabla^2\xi + \frac{1}{2}\left[(\nabla\xi)^2 + 2\frac{\partial\xi}{\partial z}\right]\frac{d}{dz_o}(\ell n U^2\rho) = \frac{g}{U^2\rho}\frac{d_\rho\xi}{dz_o} \qquad (3)$$

which is clearly a nonlinear equation if U and ρ are arbitrary. However, if $U^2(z_o)\rho(z_o)$ and $d\rho/dz_o$ are both constant then (3) reduces to

$$\nabla^2\xi + \sigma^2\xi = 0 \qquad (4)$$

where $\sigma^2 = g|d\rho/dz_o|/\rho U^2$ = constant also. However in the Boussinesq approximation subject to which (3) was derived, $\sigma^2 = -N^2/U^2$ and (4) may be re-written, after Fourier transformation of the x-dependence, as:

$$\frac{\partial^2\xi}{\partial z^2} - \left(\frac{N^2}{U^2} - k_x^2\right)\xi = 0 \qquad (5)$$

which is exactly the same wave equation which arises by application of linear perturbation theory, thus demonstrating that for mean flows with constant N and U, linear theory delivers exact nonlinear solutions to the problem but for a topography which is determined a-posteriori as that discussed previously with respect to Figure 2. This suggests a simple iterative technique with which exact nonlinear solutions for any desired topography may be simply constructed. What one does is simply to continue to adjust the topography employed in the linear calculation until the "actual" topography coincides with that desired. One example from such a construction is shown in Figure (3) which is from Laprise and Peltier (1986) in which plate (a) is for a flow which is essentially hydrostatic and plate (b) one for which non-hydrostatic effects have begun to become important.

This construction may be employed to compute the dependence of the critical Froude number $Fr_c = Nh_c/U$ upon the second nondimensional parameter Na/U which is the non-dimensional mountain half-width. Clearly, in the limit Na/U -> ∞ the long wave hydrostatic approximation must apply, in which limit Miles and Huppert (1969) have shown for symmetric topography $z_s = a^2h/(x^2 + a^2)$ that the critical Froude number $Fr_c = 0.85$. Figure (4),

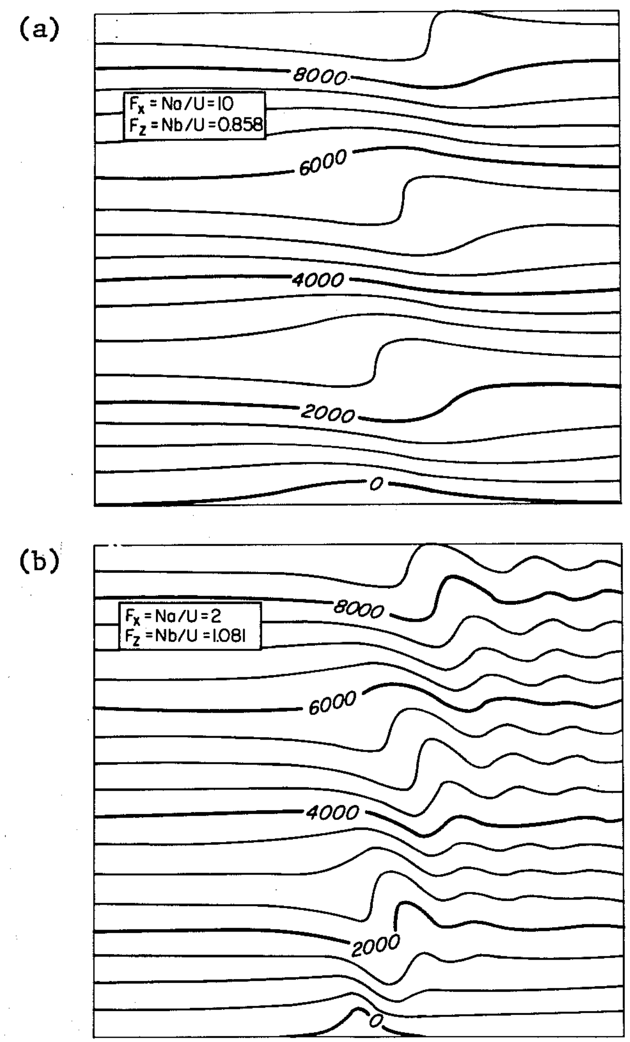

Figure 3a: Exact nonlinear solution for the stream function using iteration to match the correct lower boundary condition. The parameters are as in Figure 1 with $2\pi/N$ = 10.2 min and U = 4 m s^{-1}. Here the topography is the symmetric $z_s(x) = a^2b/(x^2+a^2)$ with Na/U = 10 so that the flow is hydrostatic and Nb/U = 0.858 so that it is also supercritical.

Figure 3b: Same as Figure (3a) but with Na/U = 2 in which case the flow is strongly nonhydrostatic and Nb/U = 1.081 so that it is also supercritical.

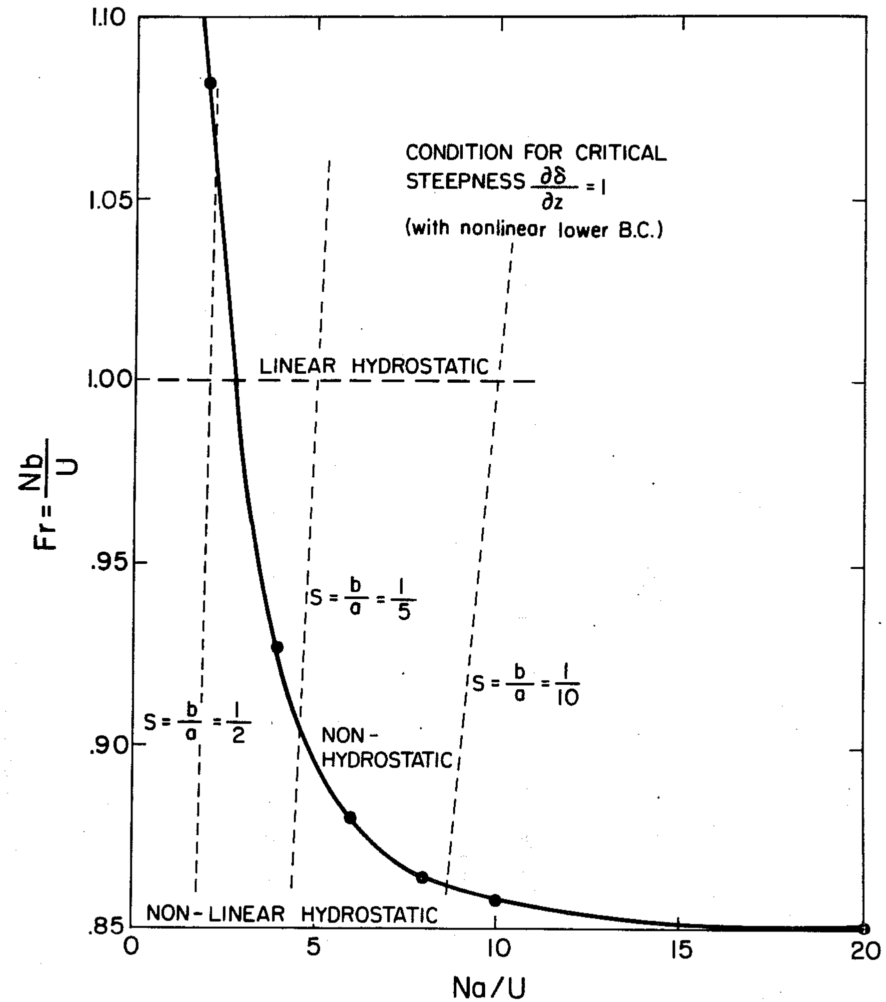

Figure 4: Variation of the critical Froude number with non-dimensional mountain half-width from the exact non-hydrostatic Long's model.

from Laprise and Peltier (1986), shows the dependence of this critical Froude number on Na/U for the general nonhydrostatic problem which demonstrates that the asymptotic (hydrostatic) condition essentially obtains for Na/U > 10 whereas for Na/U < 5 non-hydrostatic effects are large.

4. SUPERCRITICAL FLOWS WITH CONSTANT N AND U: THE SATURATION CONCEPT

The above described technique for the construction of nonlinear steady solutions, for flows with constant N and U, is of course based on the assumption that such a steady state solution exists. We might reasonably ask, however, whether any such steady state solution existsin the supercritical states in which Long's model predicts that streamlines overturn. We might furthermore speculate on the events which would occur in a flow which was forced to enter the supercritical regime. A detailed numerical analysis of what does in fact occur in such circumstances was presented in Peltier and Clark (1983), amplifying considerably the discussion in Clark and Peltier (1977). In terms of wave drag the result is clearly summarized in Figure (5) which shows surface wave drag $D_w(0)$ as a function of time for two experiments (numbered 34 and 36) which differ from one-another only in terms of vertical spatial resolution (34 has 97 vertical grid points and Δz = 153.6 m and 36 has 192 vertical grid points and Δz = 79.3 m: both experiments have 136 grid points in the horizontal with Δx = 58.9 m). In these experiments, which both are for topography $z_s(x) = a^2h/(x^2 + a^2)$ with a = 3 km, h = 400 m, and N = 0.99 x 10^{-2}-s^{-1}, the flow is slowly accelerated to a speed of 5 m s^{-1} for which case the Froude number is Nh/U = 0.792 < 0.85, so that the flow is subcritical. After about 150 min the mean flow is decelerated to U = 4 m s^{-1} in which case Nh/U = 0.9905 which is supercritical. Figure (5) shows that after deceleration the wave drag first falls and then rises dramatically and in a quasi-linear fashion. According to Long's steady state model, results of which are also

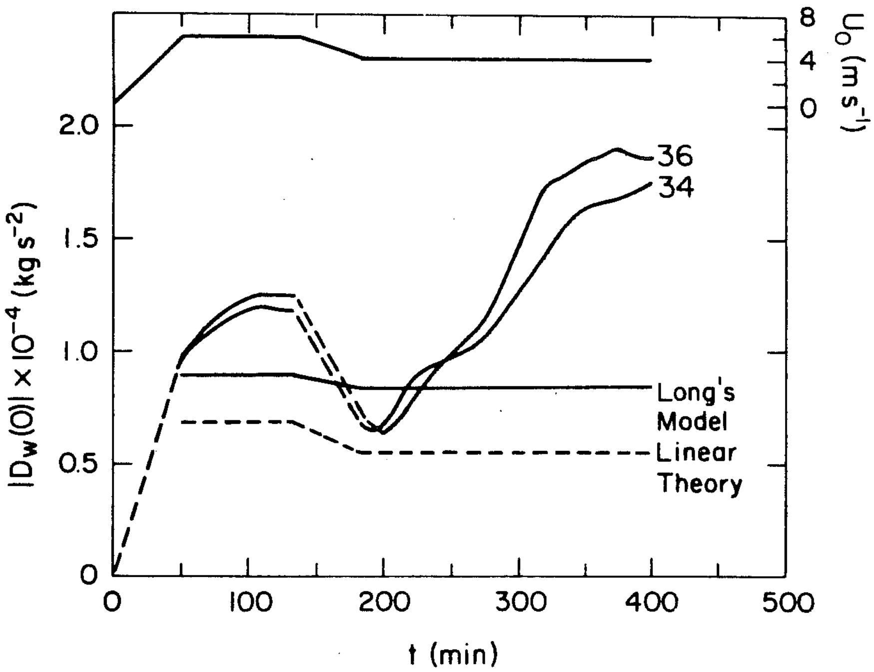

Figure 5: Surface wave drag as a function of time from two nonlinear simulations which differ only in their numerical resolutions. Symmetric topography, constant upstream N and U.

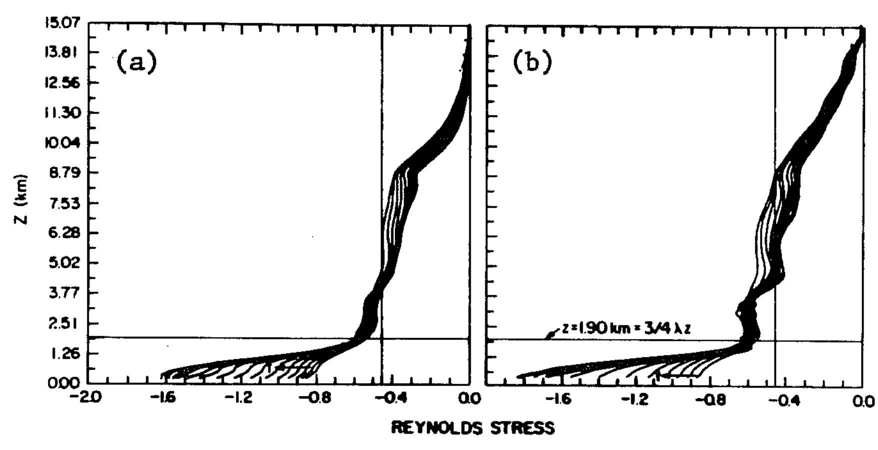

Figure 6: Time dependent Reynolds stress profiles for experiments (34) and (36) in Figure 5.

shown on Figure (5), the wave drag should fall somewhat when the flow is decelerated. In fact it rises to levels which are more than double the Long's model prediction.

Some indication as to the nature of the physical process responsible for the rapid increase of drag in the supercritical regime is provided by Figure 6 which shows profiles of Reynolds stress as a function of height through the phase of strong amplification of the wave drag. Plates (a) and (b) of this Figure are for the low and high resolution experiments respectively. The thin vertical line on each of these plates corresponds to the Reynolds stress (momentum flux) which would be associated with a wave field in which the maximum streamline steepness was precisely critical. The thin horizontal line corresponds to a height $z = 3\lambda_z/4$, which is the critical height in the wavefield at which streamlines would first be forced to exceed the critical steepness. These data clearly demonstrate that in the supercritical state the momentum flux in the wavefield above the critical height is "exactly" that which would be associated with a wave of critical steepness. No energy in excess of that associated with the critical condition is transmitted through the critical height. Rather, as is evidenced by the continuous increase of the stress in the lower levels, the "excess" is trapped in the cavity between the critical height and the surface. This is what is meant by the saturation concept. This concept is at the basis of the Lindzen-Holton (Hodges) scheme for the parameterization of wave drag. Unfortunately, in that scheme it is assumed that the excess energy and momentum are deposited into the mean flow at the height where breaking occurs. The model calculations demonstrate, on the contrary, that the excess flux is reflected, not absorbed, and it is this reflection which is responsible for the continuous amplification of the surface wave drag. As we will show in the next section, it is the low level amplification due to this cause which is at the heart of the downslope windstorm phenomenon.

5. SUPERCRITICAL FLOWS WITH VARIABLE N AND U: SEVERE DOWNSLOPE WINDSTORMS

That the above described transition mechanism plays a crucial role in the downslope windstorm phenomenon was first demonstrated in Peltier and Clark (1979) in their discussion and simulation of the storm which occurred at Boulder, Colorado on January 11, 1972. A sequence of total horizontal velocity field plots from this simulation is reproduced here as Figure 7 and the surface wave drag vs. time history of the simulation is shown in Figure 8. The first plate in Figure 7 corresponds to a time just prior to the time the wave breaks in the lower stratosphere ($t = 3200$ s or $n = 800$ in Figure 8 since the time step is 4 s) which is the time numbered 2 on Figure 8. As the drag curve enters the phase of prolonged linear amplification subsequent to breaking, the horizontal velocity in the lee is strongly amplified, eventually reaching speeds in excess of 60 m s^{-1}. The lower stratospheric wave breaking itself is seen more clearly in the evolution of the corresponding potential temperature field which is shown on Figure 9. Also evident in this simulation is the intense train of trapped lee waves which fill the domain downstream of the topography, a phenomenon which was also first simulated in the 1979 paper of Peltier and Clark.

6. THE CONCEPT OF SPONTANEOUS RESONANT AMPLIFICATION

In attempting to explain the physical process responsible for the strong amplification of the wavefield in the low levels which is observed when the critical condition is exceeded, Peltier and Clark (1979) appealed to the notion that the full wave field could be understood as a superposition of two components. The first consists of a part with structure identical to that of the wavefield which would exist if the flow consisted

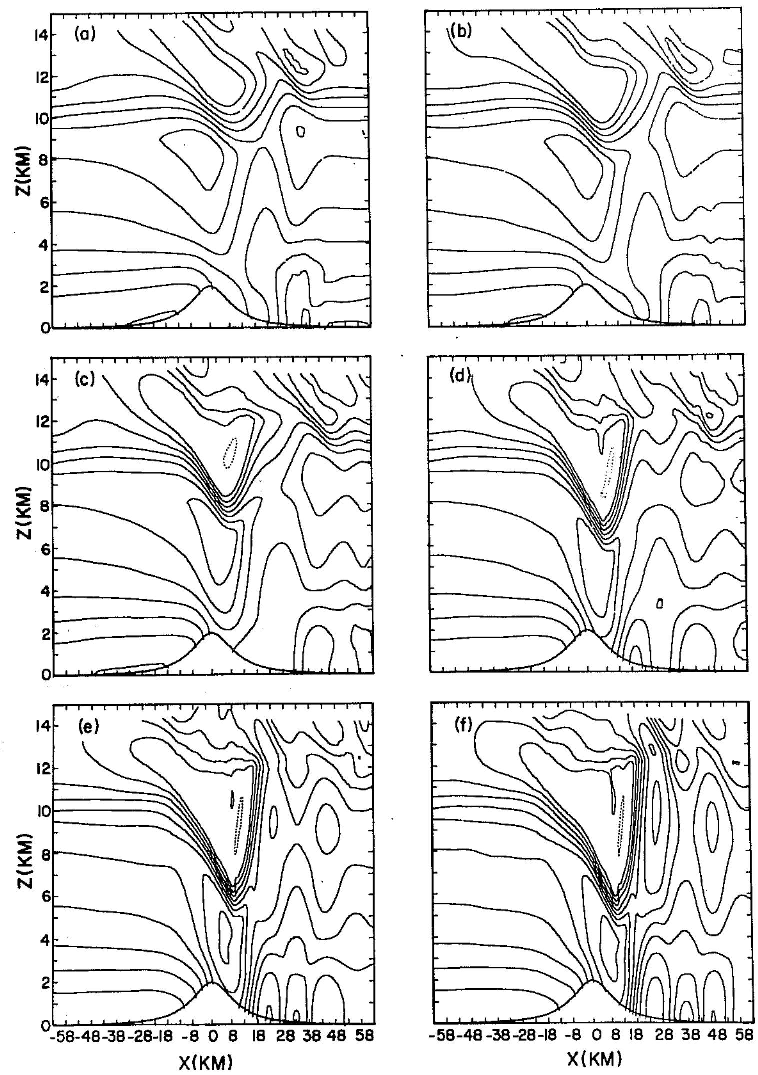

Figure 7: Total horizontal velocity as a function of time from the January 11, 1972 Boulder windstorm simulation of Peltier and Clark (1979).

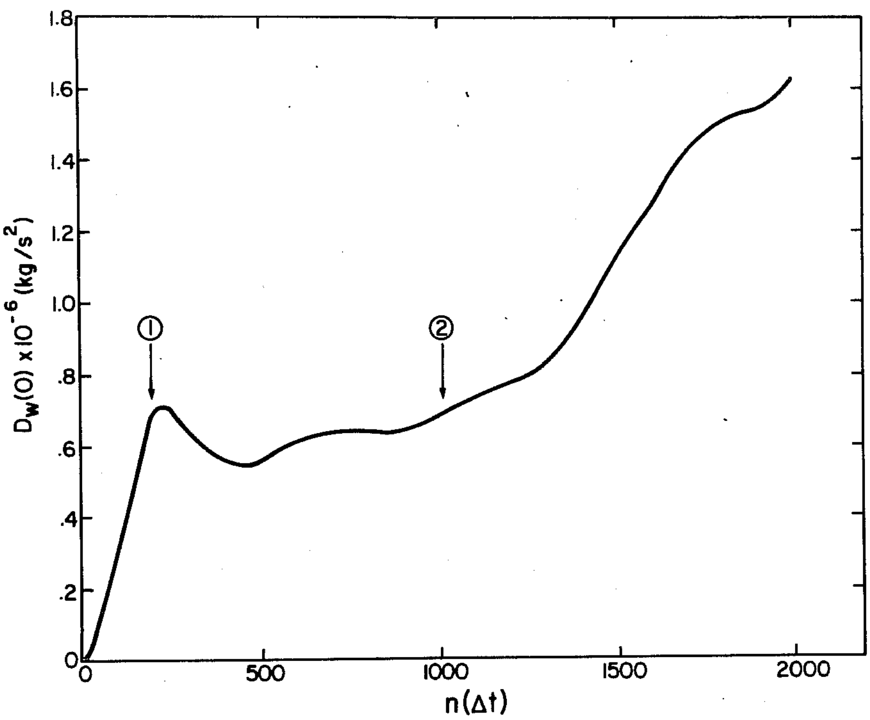

Figure 8: Surface wave drag vs. time curve from the January 11, 1972 Boulder windstorm simulation of Peltier and Clark (1979). Point 1 marks the end of the stage of model initialization while Point 2 marks the time at which the wave breaks in the lower stratosphere.

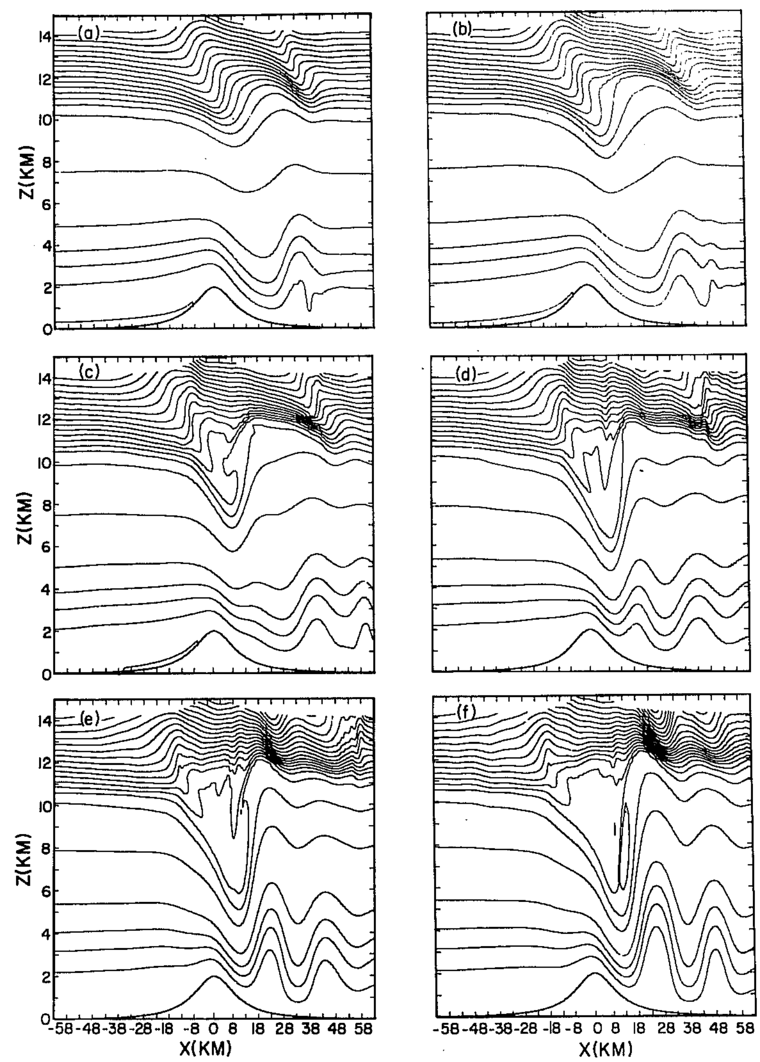

Figure 9: Potential temperature field as a function of time from the January 11, 1972 Boulder windstorm simulation of Peltier and Clark (1979).

entirely of a wave of precisely critical amplitude. The second consists of a part forced by the topography in excess of that which would be required to maintain the critical flow. This part is nonzero only in the region beneath the critical height and can be understood on the basis of the hypothesis that beyond the critical threshold the basic state wavefield supports a normal mode of instability, with growth rate $\sigma \to 0$ as $Fr \to Fr_c$, which is excited by the supercritical component of the forcing.

A very simple heuristic model as to how this would work was described in Peltier and Clark (1983) using the following linear time dependent hydrostatic internal wave equation for the density weighted perturbation vertical velocity $W = \rho w$ as:

$$\frac{\partial^4 W}{\partial z^2 \partial t^2} - k^2 N^2 W = 0 \qquad (6)$$

We may solve (6) subject to the b.c.'s (i) $w = w_o \exp(-i\,\omega t + ikx)$ on $z=0$ and (ii) perfect reflection at $z = 3\lambda_z/4$, as suggested by the results shown on Figure 6. Peltier and Clark (1983) show that (6) subject to (i) and (ii) has the solution

$$w = [at \sin(mz) + b \left(z - \frac{d}{2}\right) \cos(mx)]e^{-i\omega t} \qquad (7)$$

in the frame of reference in which the reflection occurs at the height $z=d/2$ and the ground is at $z = -d$ ($d = \lambda_z/2 = \Pi m$). Also, since we are seeking a slowly growing solution (slow on the scale of the period of the wave), the wave frequency $\omega = kU = Nk/m$. Substituting (7) into (6) we confirm the former as a valid solution if:

$$i\omega m^2 a + 2\omega^2 mb = 0 \qquad (8)$$

and if a and b are such that the lower boundary condition is satisfied. Since $\sin(mz) = 0$ on $z = -d$ the lower b.c. is satisfied if

$$b = w_o/(3d/d) \,. \qquad (9a)$$

Therefore from (8),

$$a = (2i\omega/m)\, b \ . \tag{9b}$$

From (9b) and (9a) it is clear that the growth rate a depends on the degree of supercriticality of the forcing through w_o which is the strength of the forcing on z=0 in excess of that required to maintain a "background" disturbance which is of exactly critical amplitude. Although (7) shows that w_o is not time dependent ($w_o = w\ (z=0)$), the horizontal velocity perturbation, since $u' = -(1/ik)\ (\partial w/\partial z)$, is:

$$u' = -\ (1/ik)\ [mat \cos(mz) - mb(z - \tfrac{1}{2} d)\ \sin(mz) + b\cos\ (mz)]$$

or

$$u'(z = -d) = (m\ at/ik + b/ik) \ . \tag{10}$$

Thus u' grows linearly with time and so $D_w(0) = \langle \rho_o u'w' \rangle$ will also, in accord with the results of the numerical simulation shown in Figures (5) and (8).

7. RESONANT AMPLIFICATION AND THE NONLINEAR MOUNTAIN WAVE CRITICAL LAYER

An initial test of the above described resonant amplification hypothesis was provided in Clark and Peltier (1984) wherein was described a further series of numerical simulations of nonlinear mountain waves but for mean states characterized by the presence of a wind direction reversal at some height above the surface. For these calculations wind speed was assumed to vary as $U(z) = U_o \tanh[(z - z_i)/b]$ with $U_o = 8\ m\ s^{-1}$, b = 600 m, and the height of the wind reversal z_i variable. The Brunt-Vaisala frequency was fixed at $N = 0.02\ s^{-1}$. With these parameters the value of the gradient Richardson number at the critical level is $Ri_c = N^2b^2/U_o^2 = 2.25$ so that linear theory predicts that internal waves should be absorbed. This mean flow was forced with topography $z_s(x) = a^2h/(x^2 + a^2)$ with h = 300 m and a = 3 km. This implies that the Froude number at asymptotic wind speed, $Fr = Nh/U_o = 0.75$, is subcritical (< 0.85), and therefore in the absence of the sharp decrease of wind speed aloft the forced internal waves would not break. What actually occurred in these simulations is summarized by the surface drag vs. z_i data for the quasi-steady flows which obtained in the limit of long time shown here as Figure 10. On this figure the height of the wind reversal is shown non-dimensionalized with the asymptotic vertical hydrostatic wavelength of the internal wave ($\lambda_z = 2\Pi\ U_O/N$). Inspection of these data shows that the flows are characterized by extremely high drag when $z_i/\lambda_z = 0.75$ or 1.75 but by markedly lower drag than that predicted by the asymptotic Long's model solution (dashed line) when z_i/λ_z differs sufficiently from these critical values. Detailed analysis presented in Clark and Peltier shows that for z_i/λ_z close to .75 or 1.75 there is intense resonant growth of the waves in the cavity between $z = z_i$ and z=0 which is strikingly similar to that observed in flows with U(z) constant when the waves were forced to break. On this basis it was argued that the self induced resonance observed in a supercritically forced field of mountain waves occurred simply because the mean (wave deformed) state was automatically tuned when the topography was symmetric since then the first level of breaking was always at $z = 3\lambda_z/4$ as demonstrated previously in Section 2.

These data were construed by Clark and Peltier to argue that resonant growth of the waves occurred for $z_i = (3/4 + n)\ \lambda_z$ because the wave reflected from the critical level was then in phase with the incident wave when this quantization condition was satisfied. When it was not satisfied, on the other hand, the reflected wave interfered destructively with the incident wave causing the Reynold's stress in the low levels to be reduced considerably below the magnitude predicted by the appropriate asymptotic Long's model. When the system was "on resonance" the Reynolds stress profile was strongly divergent throughout the cavity between the critical level and the ground whereas when the system was "off-resonance" the stress divergence was confined to the critical level itself suggesting that in the

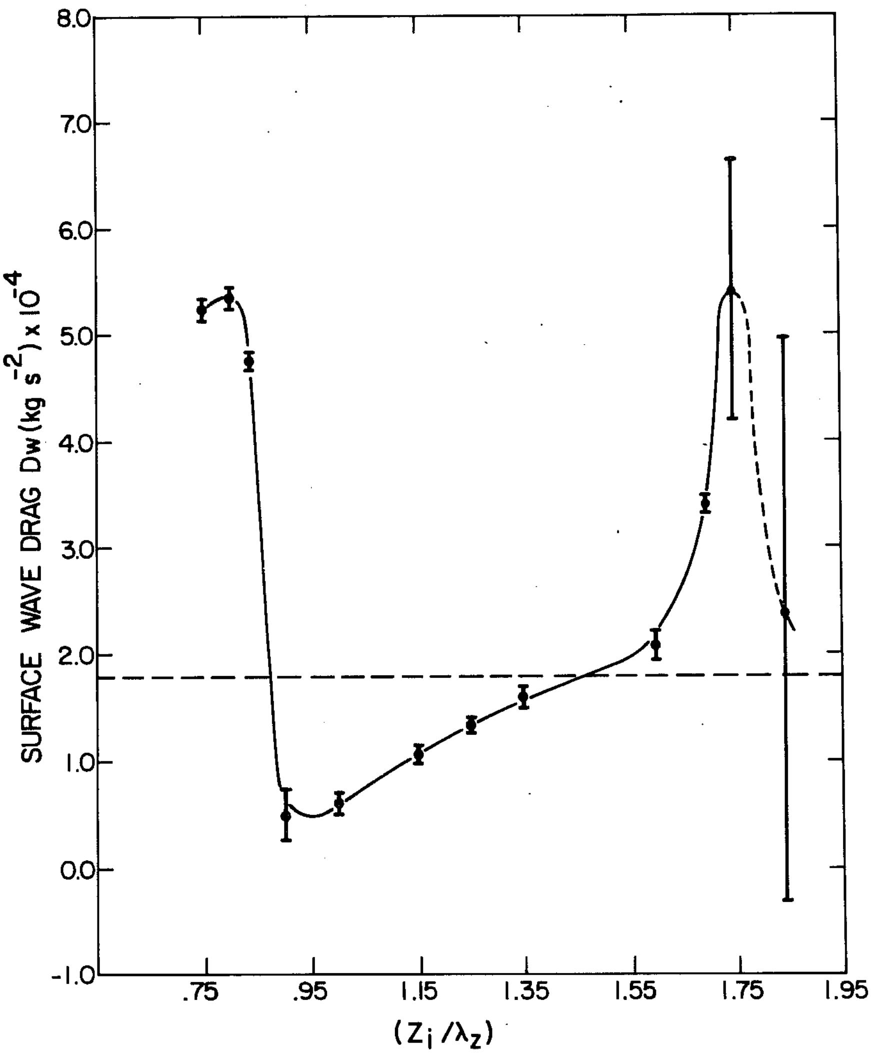

Figure 10: Surface wave drag in the limit of long time as a function of the height of the critical level above the ground from the analyses of Clark and Peltier (1984).

latter case internal wave momentum was being absorbed and the mean flow decelerated in consequence although not to the extent which would be expected in the absence of significant reflection.

It should be noted that the mechanism of wave-mean flow interaction envisioned here involves variations in the effective reflection coefficient of the "mean-flow" which depend on the phase of the incident wave, an effect whose existence clearly depends in turn upon the nonlinear interactions involved in the reflection process. For this reason one simply cannot understand this new physical process by appeal to arguments which view the critical level reflection as in any way analogous to a free surface reflection. If the latter analogy were in any sense reasonable then clearly the appropriate quantization condition would predict adjacent wave drag peaks on Figure 10 separated by half vertical wavelengths ($\lambda_z/2$) rather than full vertical wavelengths as is in fact the case. This is a consequence of the strong nonlinearity involved in the reflection process.

8. A DEFINITIVE TEST OF THE RESONANCE HYPOTHESIS

In order to finally prove or disprove the above described resonance hypothesis of the origin of downslope windstorms we are clearly obliged to give greater substance to the assumptions which underly it. The most important of these is clearly the notion that as the mountain wave field is caused to exceed critical steepness at $z = 3\lambda_z/4$, then the region of overturned streamlines begins to act as a reflector of all wave energy in excess of that required to establish the critical condition itself. This is an hypothesis which is clearly amenable to direct test in principal. All we would need to do to establish it, is simply to show that as the critical Froude number is exceeded the wavefield predicted by Long's model becomes unstable to a mode of instability trapped between the ground and the wave induced critical region. We could then compare the perturbation stream function of this mode with difference stream functions computed from the nonlinear model through the period of low level Reynolds stress amplification. If these fields were the same then the resonance hypothesis would be firmly established.

Obviously the above suggested test of the resonance hypothesis would involve an analysis of the stability of a complicated two dimensional nonlinear flow over an irregular lower boundary. This is a nontrivial technical problem. In fact, however, a formalism has recently been developed with which such a calculation can be performed. This formalism, which may be used to solve any two dimensional non-separable boundary value problem, has been employed by Klaassen and Peltier (1985a,b) to investigate the problem of the transition to turbulence in a field of nonlinear Kelvin-Helmholtz waves and by Moore and Peltier (1986a,b) to investigate the problem of the origin of frontal cyclones. The same methods have now been applied by Laprise and Peltier (1986a,b) to analyse the stability of the nonlinear field of internal waves generated by Long's model. Without going into any detail at all in the description of the results which have very recently been obtained in this work, suffice it to say that this analysis has fully confirmed the validity of the basic assumption underlying the resonance hypothesis. Just as the critical Froude number which characterizes the steady state wavefield exceeds the critical value, the flow becomes unstable to a single trapped mode whose growth rate increases from zero as the Froude number is further increased. We see linear rather than exponential low level growth in the windstorm simulations simply because, immediately as the critical condition is established in the time dependent model, this region in the fluid begins to reflect the excess energy in the wavefield. The structure which grows is therefore that of the mode with zero growth rate which is the one which exists at $Fr = Fr_c$ and it is this structure which plays the role of the eigenfunction of the 1-D harmonic oscillator in the schematic theory of Section 6.

9. CONCLUSIONS

The nature of the wave-mean flow interaction which occurs in a field of large amplitude topographically forced internal waves bares no resemblance at all to the interaction predicted by linear theory. Strong interaction occurs only when the waves break and appears to be characterized, at least for symmetric topography, by a reflection of the "supercritical" component of the wave from the level of breaking. In such circumstances the deposition of momentum which decelerates the mean flow is apparently not restricted to the height at which breaking occurs but rather is distributed throughout the fluid from the breaking level to the mountain crest. Below the mountain crest and in the lee of the topography, on the other hand, the mean flow is simultaneously accelerated and a low level "jet" forms which has maximum intensity in the immediate lee of the topography where it explains the downslope windstorm phenomenon. Therefore, although the saturation hypothesis of Lindzen-Holton (Hodges) is nicely verified by the nonlinear time dependent simulations of the nonlinear flow, the assumption that the excess momentum is simply deposited into the mean flow at the breaking level has been shown to be erroneous. This clearly has rather important potential implications for the problem of wave drag parameterization, since it shows that although breaking usually occurs in the lower stratosphere, the wave-mean flow interaction will extend throughout the troposphere. In any event, the schemes which are currently being employed in general circulation models to parameterize the influence of mountain wave drag will have to be modified considerably to represent the actual hydrodynamic processes which our simulations have revealed to occur.

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