DYNAMICALLY STABLE NONLINEAR STRUCTURES

C. E. Leith
Courant Institute of Mathematical Sciences
New York University, New York, USA

1. INTRODUCTION

In recent years there has been increasing interest in the possibility that there may be dynamically stable nonlinear structures, such as solitons or modons, embedded in the otherwise quasi-two dimensional turbulent flow characterizing the large-scale behavior of the atmosphere. This possibility has important implications for predictability and prediction. It has been suggested, for example, by McWilliams (1980) that dipole blocking structures in the atmosphere may be modons.

Solitons and modons are special localized solutions of the nonlinear dynamics equations. Two aspects of these equations -- nonlinear interaction and linear dispersion -- might be expected to destroy any local structure. Together, however, they can balance each other and preserve certain structures in a surprisingly stable way.

Long (1964) found soliton solutions in a β-channel and Benney (1966) carried through a detailed perturbation calculation to derive a Korteweg-deVries (KdV) equation appropriate for Rossby waves. Solitons are exact solutions of the KdV equation, but the KdV equation is only an approximation to, say, the barotropic vorticity equation and is only formally valid for weak dispersion and weak nonlinearity. The KdV equation describes dependence in only one space dimension taken as longitude in these applications. Latitudinal dependence is determined by the constraints of meridional boundary conditions. Redekopp (1977) has worked out the theory of Rossby solitons in considerable detail.

Linear dispersion can also be induced by effects of bottom topography in a shallow water model of the ocean. The corresponding solitons have been examined for stability and practical numerical simulation by Malanotte Rizzoli (1980). In her simulations, she finds that soliton-like solutions persist and appear to be robust for conditions that greatly exceed the formal requirements of the perturbation theory.

An alternate construction of a localized solution of the barotropic vorticity equation was provided by Stern (1975). His modon solution is a dipole confined within a circle that is stationary with respect to uniform zonal flow. Modons are exact solutions rather than perturbation approximations but suffer from discontinuities at the circle boundary. Larichev and Reznik (1976) generalized Stern's modon and weakened the boundary discontinuity by attaching an exterior solution that decayed sufficiently rapidly to preserve the local nature of the modon. Such modons move with respect to a uniform zonal flow. Flierl et al. (1980) have generalized modons still further to equivalent barotropic and baroclinic cases and have shown that, having once constructed a dipole modon, monopole riders of great variety may be added. McWilliams (1980) has matched the parameters of an equivalent barotropic modon roughly to the observed characteristic of a dipole atmosphere blocking event observed over the North Atlantic Ocean in January 1963.

I shall describe in detail the construction of an equivalent barotropic modon in Section 2 and of its riders in Section 3. The spectral consequences of a rider vorticity discontinuity are examined in Section 4, and finally in Section 5 I shall summarize the results of some numerical modon experiments.

2. MODON STRUCTURE

We shall describe a localized modon solution for the equivalent barotropic vorticity equation

$$(\nabla^2\psi-\alpha^2\psi)_t + \beta\psi_x + J(\psi,\nabla^2\psi) = 0 \qquad (2.1)$$

which determines the evolution of the stream function ψ for the flow of shallow water of mean depth h on a β-plane with Coriolis coefficient $f = f_0 + \beta y$. Here α is the deformation wavenumber with

$$\alpha^2 = \frac{f_0^2}{gh} \quad . \qquad (2.2)$$

and g is an equivalent gravitational acceleration such that $(gh)^{\frac{1}{2}}$ is the speed of gravity waves. The Jacobian here is defined in the usual way as

$$J(\psi,\phi) = \psi_x\,\phi_y - \psi_y\,\phi_x \qquad (2.3)$$

There exists in this case a potential vorticity

$$Z = f + \nabla^2\psi - \alpha^2\psi \qquad (2.4)$$

in terms of which Eq. (2.1) may be rewritten

$$Z_t + J(\psi,Z) = 0 \qquad (2.5)$$

displaying Z as conserved following the flow.

The linear Rossby wave solutions of Eq. (2.1) are given by eigenfunctions of ∇^2 such that

$$\nabla^2\psi = -\lambda^2\psi \tag{2.6}$$

For these the Jacobian term vanishes and Eq. (2.1) reduces to the linear equation

$$-(\alpha^2+\lambda^2)\psi_t + \beta\psi_x = 0 \tag{2.7}$$

describing waves propagating in the x-direction with velocity

$$c = -\frac{\beta}{\alpha^2+\lambda^2} \tag{2.8}$$

Since $0 < \lambda^2 < \infty$, c is bounded with $-\beta/\alpha^2 < c < 0$. Rossby waves are oscillatory in space like $\sin \lambda x$ and are not therefore localized solutions.

A localized solution must drop off rapidly away from some central region. As an outer solution with this property we take another eigenfunction of ∇^2 but one such that

$$\nabla^2\psi = \mu^2\psi \tag{2.9}$$

In particular we choose

$$\psi = A\, K_1(\mu r)\, \sin\theta \tag{2.10}$$

where K_1 is the modified Bessel function of the second kind of order 1. Again in the outer region the Jacobian vanishes and Eq. (2.1) reduces to Eq. (2.7) but with λ^2 replaced by $-\mu^2$. The outer solution (2.10) propagates therefore in the x-direction with velocity

$$c = -\frac{\beta}{\alpha^2-\mu^2} \tag{2.11}$$

Since $0 < \mu^2 < \infty$, the range of possible c values for localized solutions is $-\infty < c < -\beta/\alpha^2$ and $0 < c < \infty$, disjoint from the possible Rossby wave velocities of Eq. (2.8). In Eq. (2.10) r and θ are polar coordinates in a moving frame with, say,

$$r^2 = (x-ct)^2 + y^2$$
$$\sin\theta = y/r \tag{2.12}$$

To avoid the singularity in ψ at $r = 0$ given by Eq. (2.10), we introduce a smooth inner solution which we let join the outer one at a circle of radius $r = a$. We take as the inner solution for $r \leq a$

$$\psi = BJ_1(\lambda r)\, \sin\theta - C\, r\, \sin\theta \tag{2.13}$$

where J_1 is the Bessel function of order 1.

The first term is again an eigenfunction of ∇^2 satisfying Eq. (2.6) and would by itself propagate in the x-direction with a velocity given by Eq. (2.8). The second term, however, introduces a constant advecting velocity C. In order that the inner and outer propagation velocities be the same we must impose a velocity constraint

$$C = \beta \left[\frac{1}{\lambda^2} - \frac{\alpha^2 + \lambda^2}{\lambda^2(\alpha^2 - \mu^2)} \right] \tag{2.14}$$

that determines the coefficient C for any choice of inner and outer wavenumbers, λ and μ .

We match the inner and outer solutions at $r = a$ by imposing as many continuity conditions as possible. From the continuity of ψ and ψ_r at $r = a$ we have

$$A\, K_1(\mu a) = B\, J_1(\lambda a) - Ca \,, \tag{2.15}$$

and

$$A\, \mu a\, K_1'(\mu a) = B \lambda a\, J_1'(\lambda a) - Ca \tag{2.16}$$

Let $\lambda a = \bar{\lambda}$, $\mu a = \bar{\mu}$. By subtraction we may eliminate the term Ca and find

$$A\, [\, K_1(\bar{\mu}) - \bar{\mu} K_1'(\bar{\mu})\,] = B[J_1(\lambda) - \bar{\lambda}\, J_1'(\bar{\lambda})\,] \tag{2.17}$$

Recursion relations for Bessel functions permit Eq. (2.17) to be written in simpler form as

$$A\, [\bar{\mu} K_2(\bar{\mu})\,] = B[\, \bar{\lambda} J_2(\bar{\lambda})\,] \tag{2.18}$$

whence

$$\begin{aligned} A &= S[\, \bar{\mu}\, K_2(\bar{\mu})\,]^{-1} \\ B &= S[\, \bar{\lambda}\, J_2(\bar{\lambda})\,]^{-1} \end{aligned} \tag{2.19}$$

The coefficient S is determined by Eq. (2.15) to be

$$S = Ca \left[\frac{J_1(\bar{\lambda})}{\bar{\lambda} J_2(\bar{\lambda})} - \frac{K_1(\bar{\mu})}{\bar{\mu} K_2(\bar{\mu})} \right]^{-1} \tag{2.20}$$

The conditions imposed so far suffice to determine the amplitude coefficients A, B, and C for any choice of radius a and wavenumbers λ and μ . By Eq. (2.11) a choice of μ is equivalent to a choice of the overall propagation velocity c. Then the choice of λ determines C by Eq. (2.14). If we next choose a radius a then $\bar{\lambda}$ and $\bar{\mu}$ are defined, S is determined by Eq. (2.20) and finally A and B by Eqs. (2.19).

The most important continuity conditions have been satisfied, but we still have the freedom to choose λ for a given value of μ in such a way that the vorticity $\zeta = \nabla^2\psi$ is also continuous at $r = a$.

The continuity condition for ζ at $r = a$ is

$$A\,\bar{\mu}^{2}\,K_1(\bar{\mu}) = -B\,\bar{\lambda}^{2}\,J_1(\bar{\lambda}) \tag{2.21}$$

which may be combined with Eqs. (2.19) to give

$$\frac{\bar{\lambda}J_1(\bar{\lambda})}{J_2(\bar{\lambda})} = -\,\frac{\bar{\mu}K_1(\bar{\mu})}{K_2(\bar{\mu})} \tag{2.22}$$

For any value of $\bar{\mu}$ the expression on the right is well defined and negative. Thus λ must be in those intervals of the $\bar{\lambda}$ range where J_1 and J_2 have opposite sign. We shall consider only the gravest such interval $(j_1^{(1)}, j_2^{(1)})$ where $\bar{\lambda}$ is smallest and the inner solution has the smoothest structure. The solid curve in Fig. 1 shows the mapping $\bar{\mu} \rightarrow \bar{\lambda}$ into this interval given by Eq. (2.22).

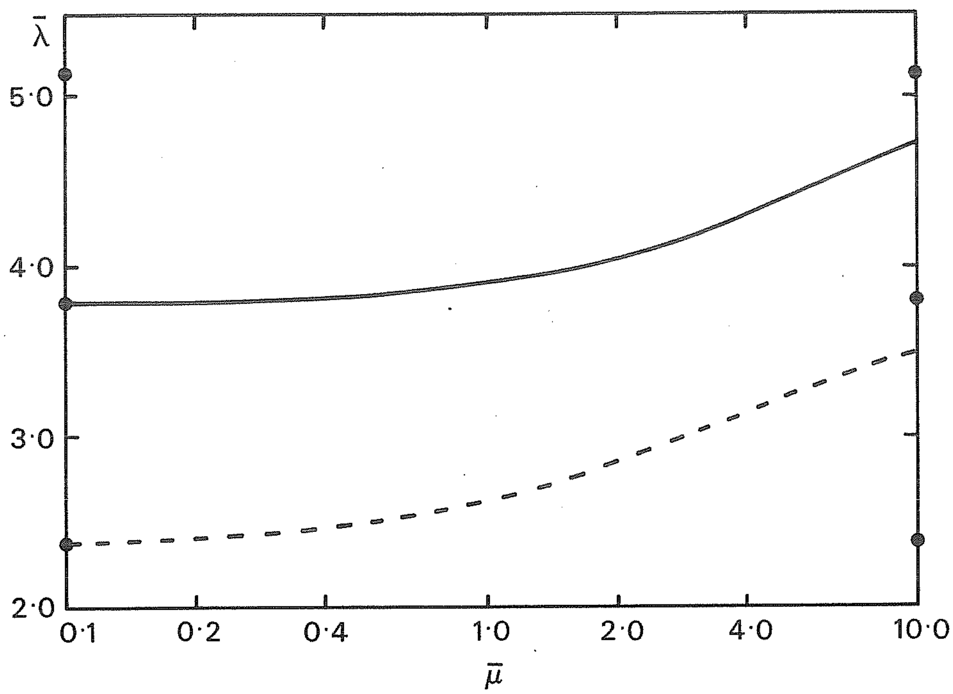

Fig. 1. Inner wavenumber $\bar{\lambda}$ vs. outer wavenumber $\bar{\mu}$ satisfying vorticity continuity conditions for a modon (solid) and a rider (dashed). Dots on the $\bar{\lambda}$ - axis delimit solution intervals.

The modon so constructed is a localized vorticity dipole with an amplitude determined by its radius a and its velocity c. The required form of the second term in Eq. (2.13) imposes the dipole structure on the first term and on Eq. (2.10).

3. RIDER STRUCTURE

Once a modon has been constructed it is possible to add a rider to it and still have a localized propagating solution of Eq. (2.1). The simplest rider is a vorticity monopole. Let the rider stream function be indicated by ψ' to distinguish it from that of the modon; the total stream function will then be $\psi + \psi'$.

For the outer rider solution we choose a monopole solution of Eq. (2.9) with the same value of μ^2 thus for $r > a$

$$\psi' = D\ K_0(\mu r) \tag{3.1}$$

The sum $\psi + \psi'$ is still an eigenvector of ∇^2 satisfying Eq. (2.9), and the velocity c is unchanged by the rider.

We take as the inner rider solution the monopole structure

$$\psi' = E\ J_0(\lambda r) + F \tag{3.2}$$

with the same value of λ as for the modon. The first term is again an eigenfunction of ∇^2 satisfying Eq. (2.6). Since the constant F does not affect velocities, the velocity constraint of Eq. (2.14) is still appropriate and satisfied.

We match the inner and outer rider solutions at r = a by imposing continuity conditions as for the modon. We find from continuity of ψ' and ψ'_r that

$$D = R\ [\bar{\mu}K_1(\bar{\mu})]^{-1}$$
$$E = R\ [\bar{\lambda}J_1(\bar{\lambda})]^{-1} \tag{3.3}$$

and that

$$F = R\left[\frac{K_0(\bar{\mu})}{\bar{\mu}K_1(\bar{\mu})} - \frac{J_0(\bar{\lambda})}{\bar{\lambda}J_1(\bar{\lambda})}\right] \tag{3.4}$$

where R is an arbitrary constant unconstrained by any imposed conditions. Once a modon has been constructed a monopole rider of an approximate shape may be added with arbitrary sign and amplitude.

Continuity of rider vorticity $\zeta' = \nabla^2\ \psi'$ at r = a leads through recursion relations to the condition

$$\frac{\bar{\lambda}J_2(\bar{\lambda})}{J_1(\bar{\lambda})} = \frac{\bar{\mu}K_2(\bar{\mu})}{K_1(\bar{\mu})} \tag{3.5}$$

If this condition is satisfied then Eq. (2.20) determining the modon amplitude S becomes singular. The dashed curve in Fig. 1 shows the gravest solutions of Eq. (3.5). These must be avoided to preserve the modon structure.

A vorticity continuity condition may be imposed on the modon. It must not, however, be imposed instead on the rider, and the rider must have a vorticity discontinuity at $r = a$.

4. VORTICITY DISCONTINUITY

A discontinuity in a field tends to dominate its wavenumber spectrum at high wavenumbers and induce a characteristic power-law dependence. It is of interest to analyze this effect for the vorticity discontinuities that riders must have.

The essential aspect of the situation is revealed by considering the spectral transform of a vorticity function equal to a constant V for $r \leq a$ and vanishing for $r \leq a$. The two-dimensional fourier transform of a function $f(r)$ depending only on r may be written as

$$\hat{f}(k_1,k_2) = \int_0^\infty r\, f(r) J_0(kr)\, dr = \hat{f}(k) \tag{4.1}$$

where $k_1 = k \sin\phi$, $k_2 = k \cos\phi$ are components of a two-dimensional wave vector. The transform $\hat{f}$ depends only on k as is to be expected from symmetry.

The two-dimensional power spectrum is proportional to the square of the amplitude

$$\phi(k_1,k_2) \propto \hat{f}^2(k) \tag{4.2}$$

The one-dimensional spectrum $F(k)$ involves an integration over ϕ in wavevector space that introduces a factor k,

$$F(k) \propto k\, \hat{f}^2(k)\ . \tag{4.3}$$

In this case we have

$$\begin{aligned} f(r) &= V \text{ for } r \leq a \\ &= 0 \text{ for } r > a \end{aligned} \tag{4.4}$$

so that

$$\begin{aligned} \hat{f}(k) &= V \int_0^a r\, J_0(kr)\, dr \\ &= V\, k^{-2} \int_0^{ka} x\, J_0(x)\, dx \\ &= V\, a\, k^{-1}\, J_1(ka) \end{aligned} \tag{4.5}$$

and

$$F(k) \propto V^2 a^2\, k^{-1}\, J_1^2(ka) \tag{4.6}$$

Over the ensemble of radii and strengths this becomes

$$F(k) \propto k^{-1} \langle v^2 a^2 \; J_1^2(ka) \rangle \tag{4.7}$$

The evaluation of the bracketed average depends on knowing the joint probability distribution of v and a. In rough approximation we may assume that the average washes out the detailed oscillations of J_1^2 and leaves only its general dependence which is proportional to $(ka)^{-1}$ for ka >>1. Then the vorticity power spectrum becomes

$$F(k) \propto k^{-2} \langle v^2 a \rangle \propto k^{-2} \; , \tag{4.8}$$

and the associated kinetic energy spectrum is

$$E(k) = k^{-2} \; F(k) \propto k^{-4} \tag{4.9}$$

Such a spectral contribution of rider discontinuities to the spectrum of two-dimensional turbulent motions would not dominate the inertial range spectrum that is proportional to k^{-3}.

5. NUMERICAL STUDIES

McWilliams et al. (1981) have carried out extensive gridpoint β-plane numerical studies of barotropic modons to determine the effects of limited resolution and of dissipative processes and the resistance of modons to various levels and scale of perturbations. They find modons to be remarkably robust and not easily destroyed by perturbations. In the resolution experiments, even with only five grid intervals per modon diameter, a modon-like structure persisted although with a characteristic velocity about one-half of the theoretical value.

Should modons or modon-like structures be relevant to weather and climate simulation, then the question of required model resolution becomes of considerable interest. Gridpoint methods are notoriously poor in inducing erroneous linear dispersion with associated errors in group velocity propagation of wave packets (Grotjahn and O'Brien, 1976). The slowing down of modons at low resolution observed by McWilliams et al. (1981) is likely to be a consequence, in part, of this kind of error. I have repeated their resolution experiments with a spectral transform β-plane model in which the linear terms are treated exactly with a linear recursion operator. Fig. 2 is taken from McWilliams et al. (1981) with circled points added to show my spectral transform results

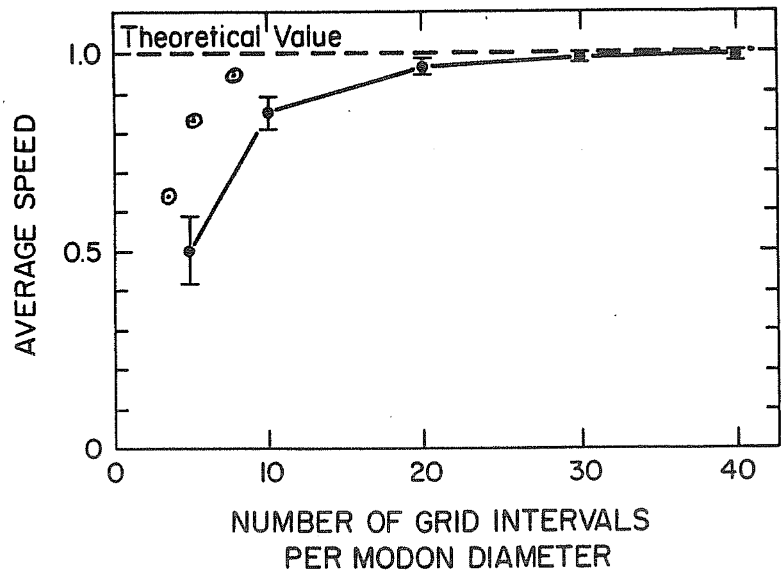

Fig. 2. Modon propagation speed as a function of resolution.

With five spectral transform grid intervals per modon diameter, the velocity is still diminished but only by 15 percent. At coarse resolution, it appears that the modon-like structure tends to enlarge and in accordance with modon dynamics to slow down. These results suggest that even low-resolution spectral transform global models, say with rhomboidal 15 truncation, should be able to treat an atmospheric blocking structure of the sort studied by McWilliams (1980).

Many blocking events do not, however, have such a simple dipole structure and may be better fit with the help of riders. An important future task, therefore, is to carry out similar numerical experiments on the properties of modons with riders.

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