On the regime of validity of sound-proof model equations for atmospheric flows

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Thanks to ...

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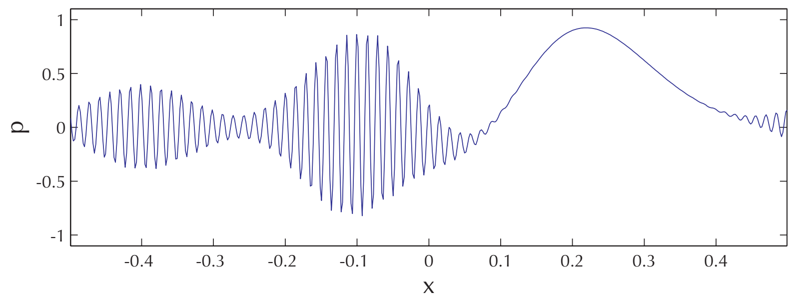
Motivation ... Numerics

Why not simply solve the full compressible equations?



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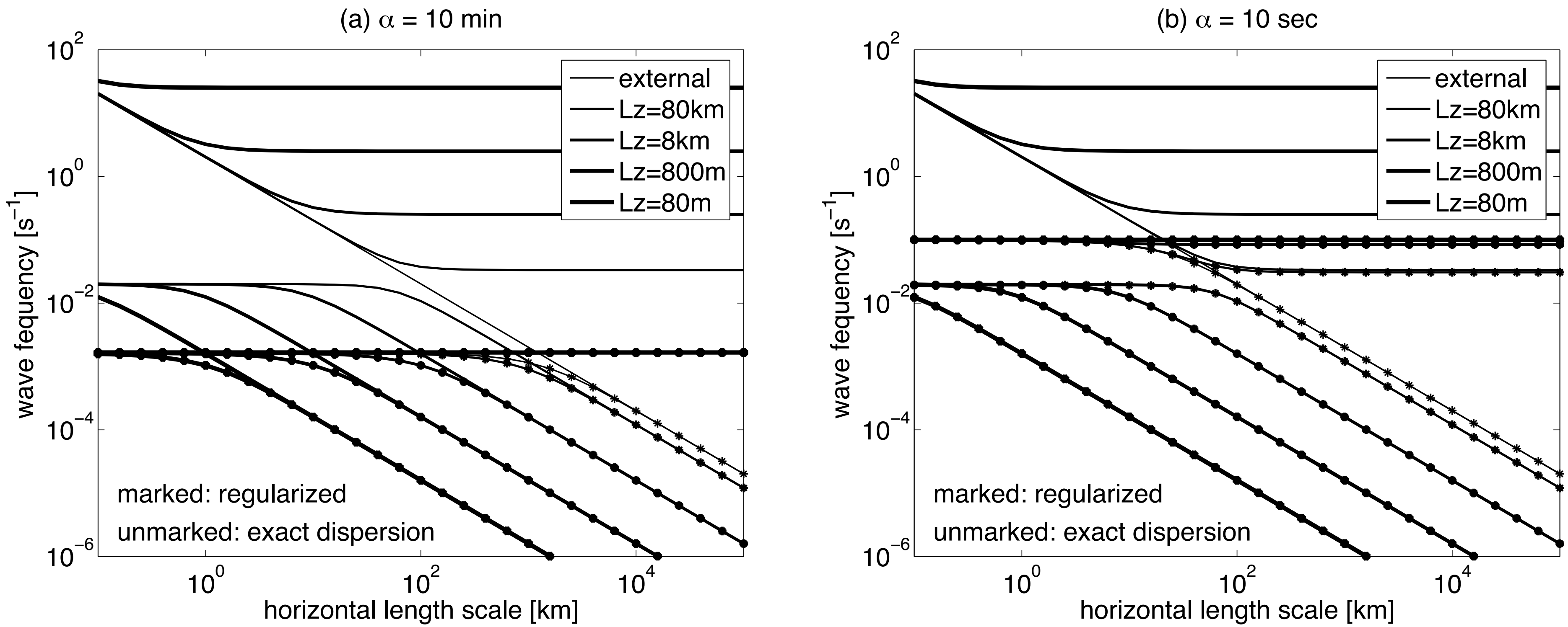


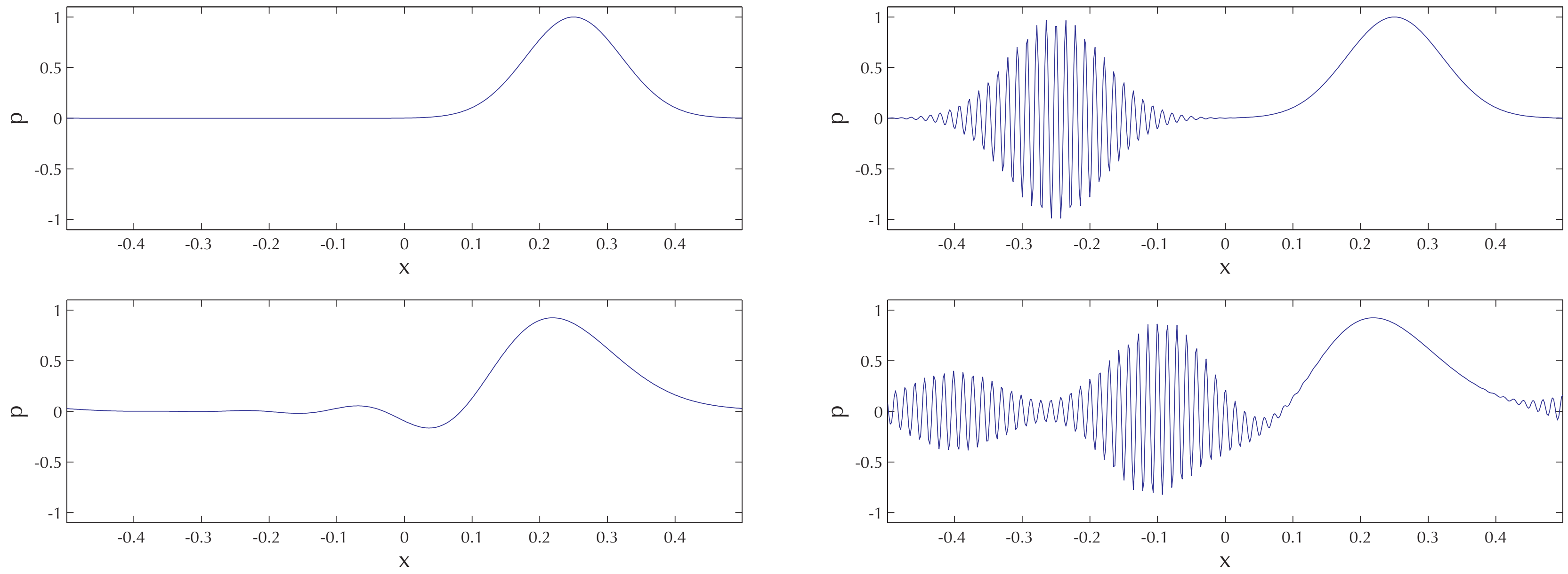
* adapted from Reich et al. (2007)

Motivation ... Numerics

Why not simply solve the full compressible equations?

Linear Acoustics, simple wave initial data, periodic domain

(integration: implicit midpoint rule, staggered grid, 512 ***grid pts.,*** $\mathrm{CFL} = 10$***)***



$t = 0$

$t = 3$

Regime(s) of validity of sound-proof models

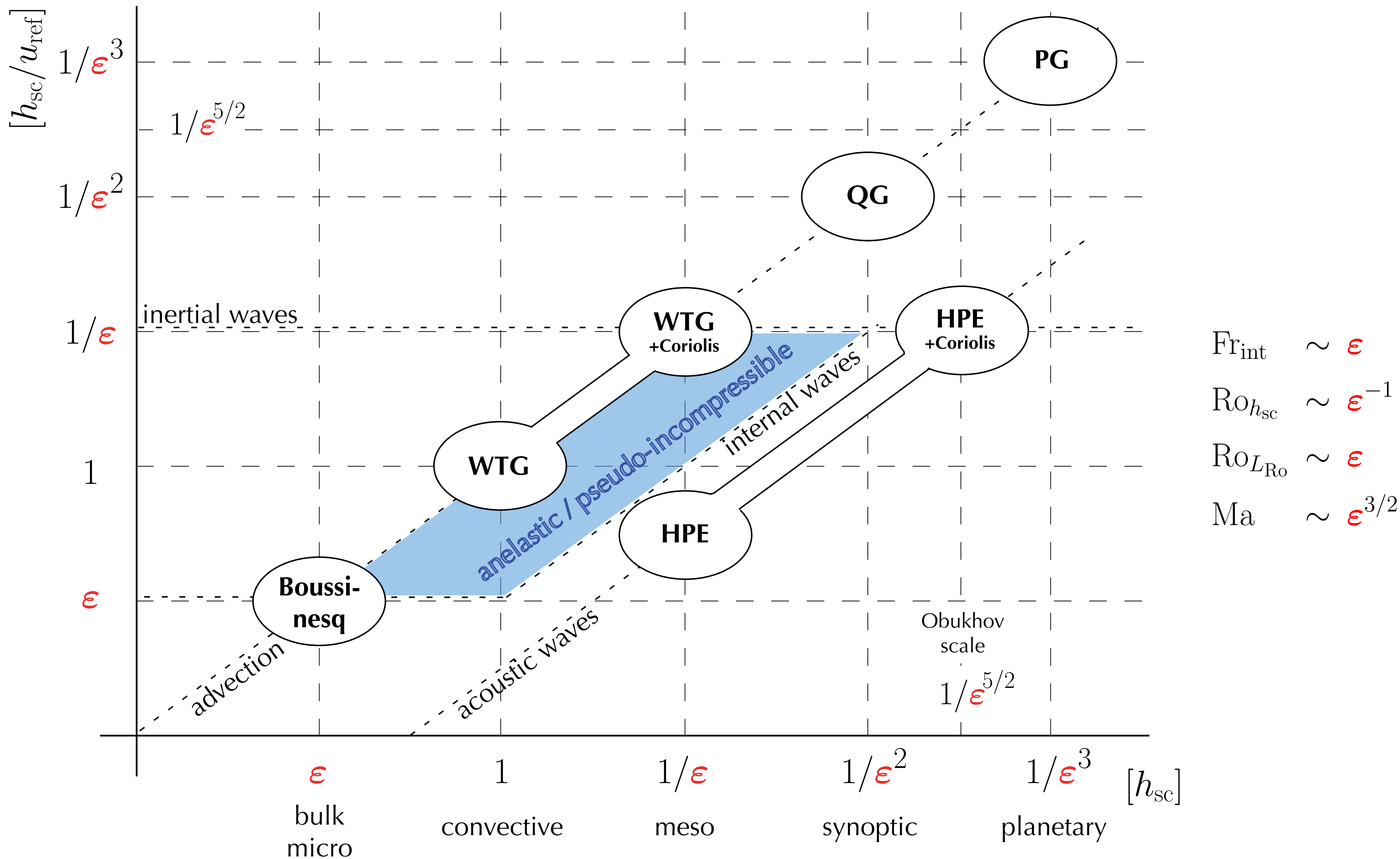
Background

Stratification limit in the design-regime

Wave-breaking regime with strong stratification

Summary

Atmospheric Flow Regimes



R.K., Ann. Rev. Fluid Mech, **42**, 2010

Sound-Proof Models

Compressible flow equations

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi = 0$$

$$(\rho w)_t + \nabla \cdot (\rho \boldsymbol{v} w) + P \pi_z = -\rho g$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0$$

drop term for:

anelastic† (approx.)

pseudo-incompressible*

hydrostatic-primitive

$$P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p / \Gamma P\,, \qquad \Gamma = c_p / R\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k} \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)$$

† e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982)

* Durran, JAS, **46**, 1453–1461 (1988)

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Parameter range & length and time scales of asymptotic validity ?

† e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982)

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Regime(s) of validity of sound-proof models

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Summary

From here on: ε is the Mach number

Regimes of Validity ... Design Regime

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Regimes of Validity ... Design Regime

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Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2)$$

* Ogura & Phillips (1962)

Regimes of Validity ... Design Regime

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$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2) \qquad \Rightarrow \qquad \Delta\overline{\theta}\,\Big|_{z=0}^{h_{\text{sc}}} < 1\ \text{K}$$

* Ogura & Phillips (1962)

Regimes of Validity ... Design Regime

Desirable:

1. **Sound-proof model** which
2. accurately represents the **(fast) internal waves**, and
3. remains accurate over **advective time scales**.

Regimes of Validity ... Design Regime

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^{\nu}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{gh_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{gh_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Realistic regime with three time scales

$$\overline{\theta} = 1 + \varepsilon^{\mu}\widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \qquad (\nu = 1 - \mu/2)$$

Regimes of Validity ... Design Regime

$$\begin{aligned}
\tilde\theta_\tau &+ \frac{1}{\varepsilon^{\nu}}\,\tilde w\,\frac{d\widehat{\theta}}{dz} & &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde\theta \\
\tilde{\boldsymbol{v}}_\tau &+ \frac{1}{\varepsilon^{\nu}}\,\frac{\tilde\theta}{\overline{\overline{\theta}}}\,\boldsymbol{k} &+ \frac{1}{\varepsilon}\,\overline{\theta}\nabla\tilde\pi &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde\theta\nabla\tilde\pi \;\cdot \\
\tilde\pi_\tau & &+ \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde w\frac{d\overline{\pi}}{dz}\right) &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde\pi - \gamma\Gamma\tilde\pi\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}$$

For the linear variable coefficient system:

- ✓ Conservation of weighted quadratic energy
- ✓ Control of time derivatives by initial data $(\tau = O(1))$

... consider internal wave scalings for $\tau = O(\varepsilon^{\nu})$:

$$\vartheta = \frac{\tau}{\varepsilon^{\nu}}\,, \qquad \pi^* = \varepsilon^{\nu-1}\tilde\pi\,,$$

Klainerman, Majda (1982) ... Dutrifoy, Majda, Schochet (2006,2008)

Regimes of Validity ... Design Regime

Fast linear compressible / pseudo-incompressible modes

$$\tilde{\theta}_{\vartheta} \;+\; \tilde{w}\,\frac{d\overline{\theta}}{dz} \;=\; 0$$

$$\tilde{\boldsymbol{v}}_{\vartheta} \;+\; \frac{\tilde{\theta}}{\overline{\theta}}\,\boldsymbol{k} + \overline{\theta}\nabla\pi^{*} \;=\; 0$$

$$\varepsilon^{\mu}\,\pi^{*}_{\vartheta} + \left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) \;=\; 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\boldsymbol{u}} \\ \tilde{w} \\ \pi^{*} \end{pmatrix}(\vartheta, \boldsymbol{x}, z) = \begin{pmatrix} \Theta^{*} \\ \boldsymbol{U}^{*} \\ W^{*} \\ \Pi^{*} \end{pmatrix}(z)\;\exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

Regimes of Validity ... Design Regime

Relation between compressible and pseudo-incompressible vertical modes

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}\frac{1}{\overline{\theta}\,\overline{P}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}\,\overline{P}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}\,\overline{P}}W^*$$

$\varepsilon^{\mu} = 0$: pseudo-incompressible case

regular Sturm-Liouville problem for internal wave modes *(rigid lid)*

$\varepsilon^{\mu} > 0$: compressible case

nonlinear Sturm-Liouville problem ...

$\frac{\omega^2/\lambda^2}{\bar{c}^2} = O(1)$: perturbations of pseudo-incompressible modes & EVals

Regimes of Validity ... Design Regime

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}\frac{1}{\bar{\theta}\,\bar{P}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\bar{\theta}\,\bar{P}}W^* = \frac{1}{\boldsymbol{\omega}^2}\frac{\lambda^2 N^2}{\bar{\theta}\,\bar{P}}W^*$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\bar{c}^2} = O(1)\right)$

- pseudo-incompressible modes/EVals $=$ compressible modes/EVals + $O(\varepsilon^{\mu})$ †
- phase errors remain small ***over advection time scales*** for $\mu > \frac{2}{3}$

The anelastic and pseudo-incompressible models remain relevant for stratifications

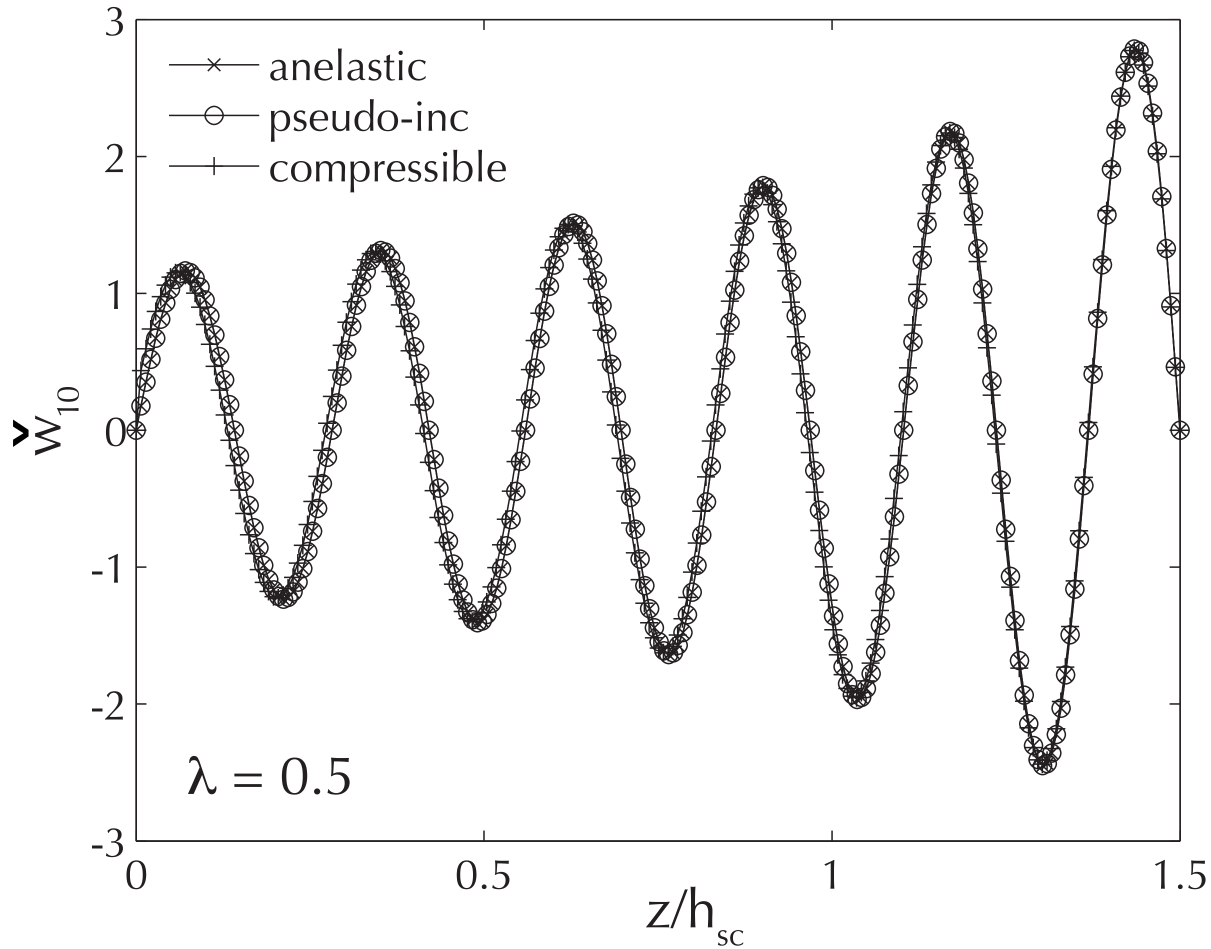
$$\frac{1}{\bar{\theta}}\frac{d\bar{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{\rm sc}} \lesssim 40\ \mathrm{K}$$

not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

† rigorous proof with D. Bresch

Regimes of Validity ... Design Regime

A typical vertical structure function $(L \sim \pi h_{\rm sc} \sim 30 \text{ km})$

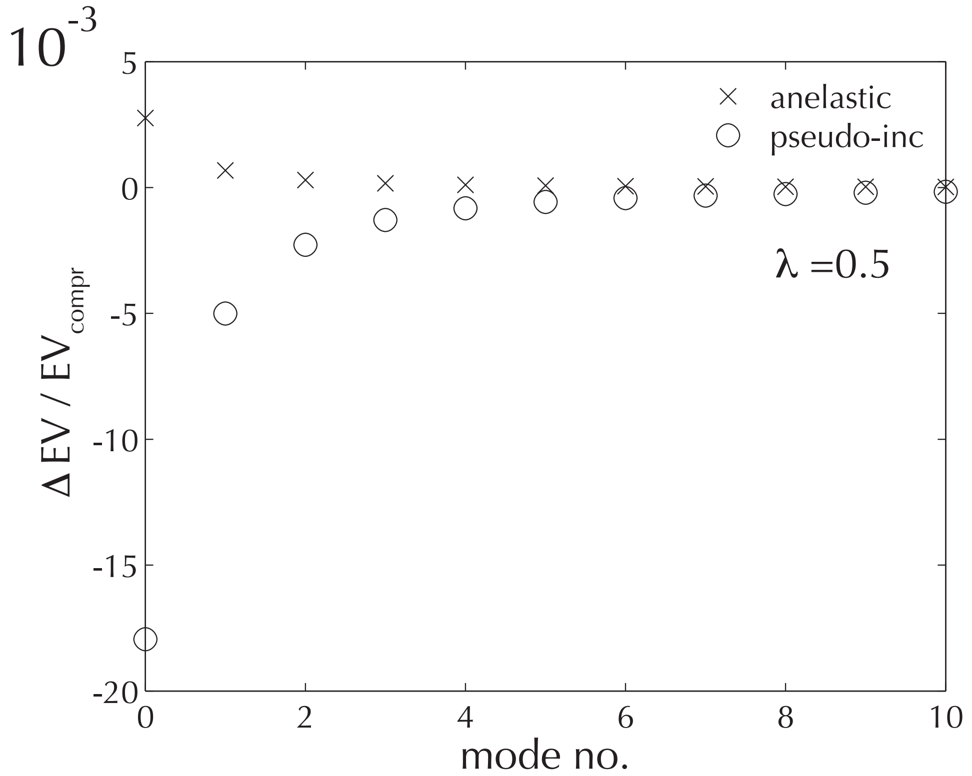


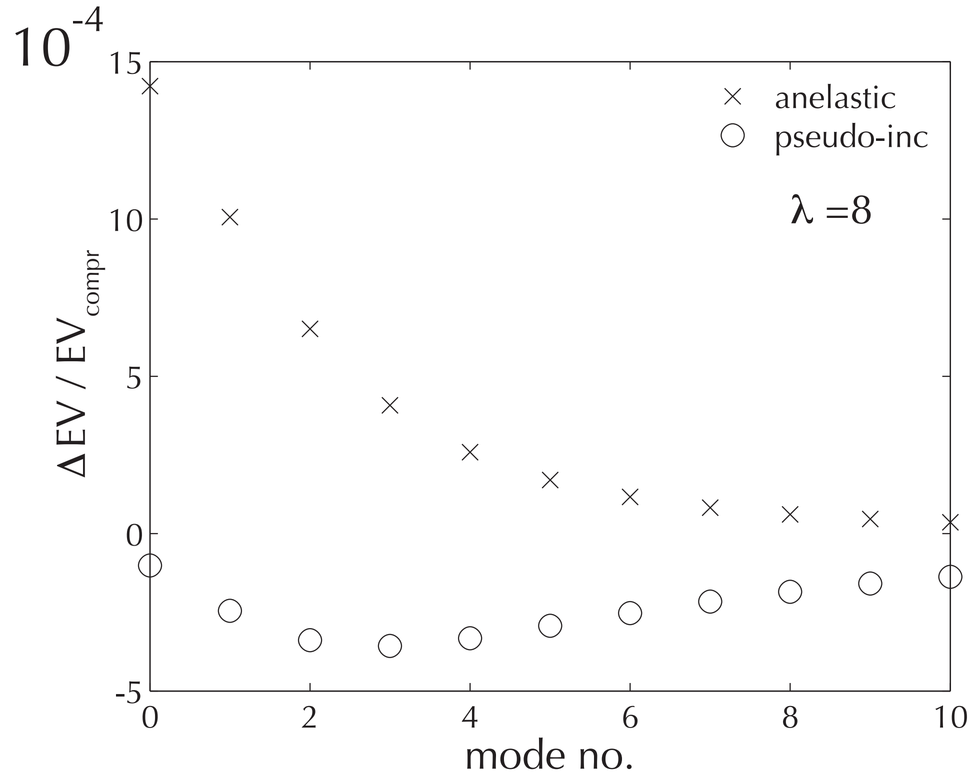
thanks to Dr. V. LeDoux, Ghent, for the SL-solver MATSLISE!

Regimes of Validity ... Design Regime

Relative eigenvalue errors

$$\frac{EV_{sproof} - EV_{compr}}{EV_{compr}}$$





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Regimes of Validity ... Design Regime

Remarks

- Estimates are uniform in the horizontal long-wave limit
  (Coriolis not yet included)

- Regime of validity includes **isothermal stratification** if

$$\frac{\gamma - 1}{\gamma} \sim \varepsilon^{2/3}$$

(Newtonian Limit)*

* K., Majda, TCFD, **20**, 525–552 (2006)

Regimes of Validity ... Design Regime

Remarks

- Estimates are uniform in the horizontal long-wave limit
  (Coriolis not yet included)

- Regime of validity includes **isothermal stratification** if

$$0.286 \approx \frac{\gamma - 1}{\gamma} \sim \varepsilon^{2/3} \approx 0.25 \qquad (\varepsilon = 1/8)$$

(Newtonian Limit)*

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Regime(s) of validity of sound-proof models

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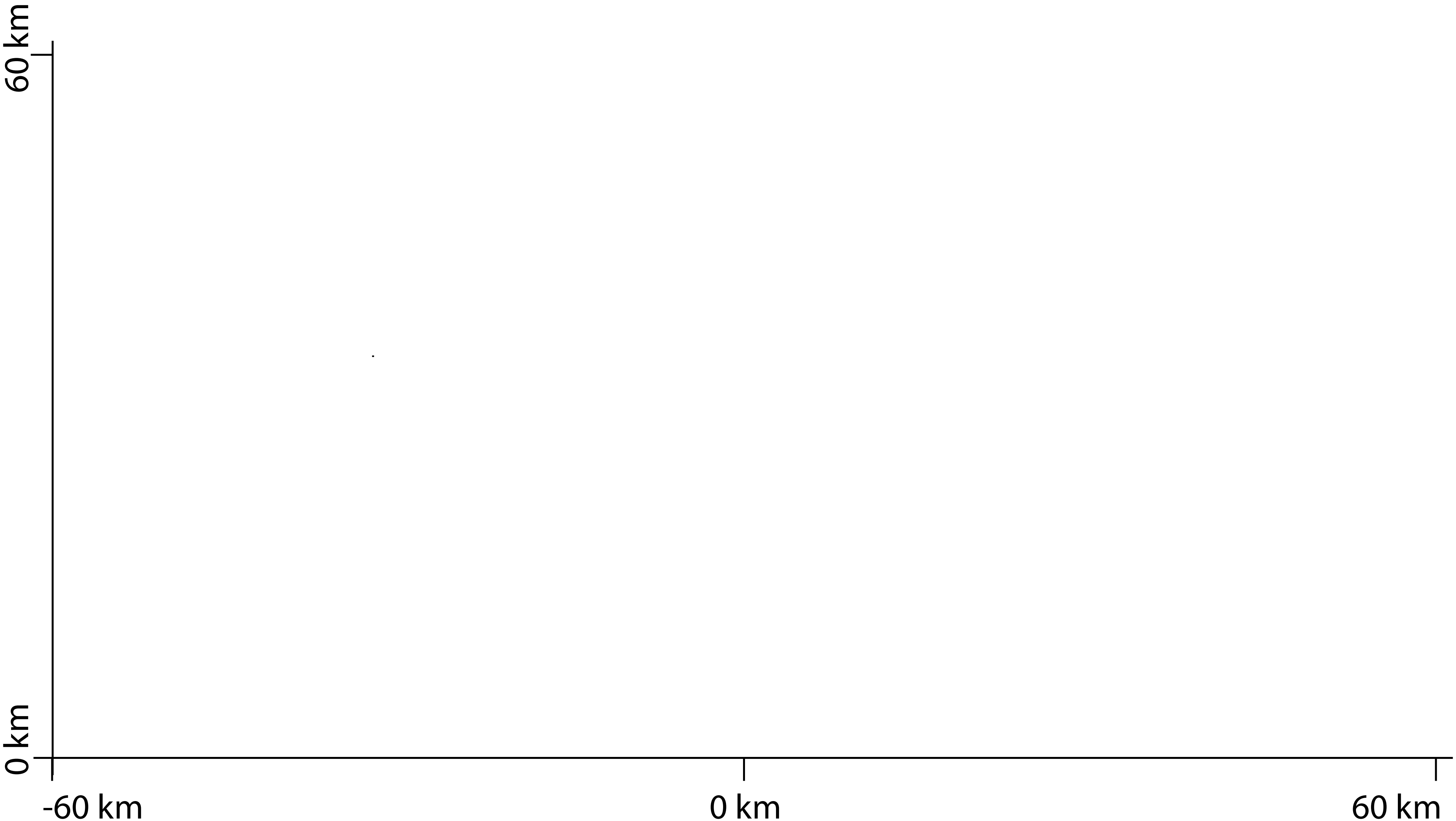
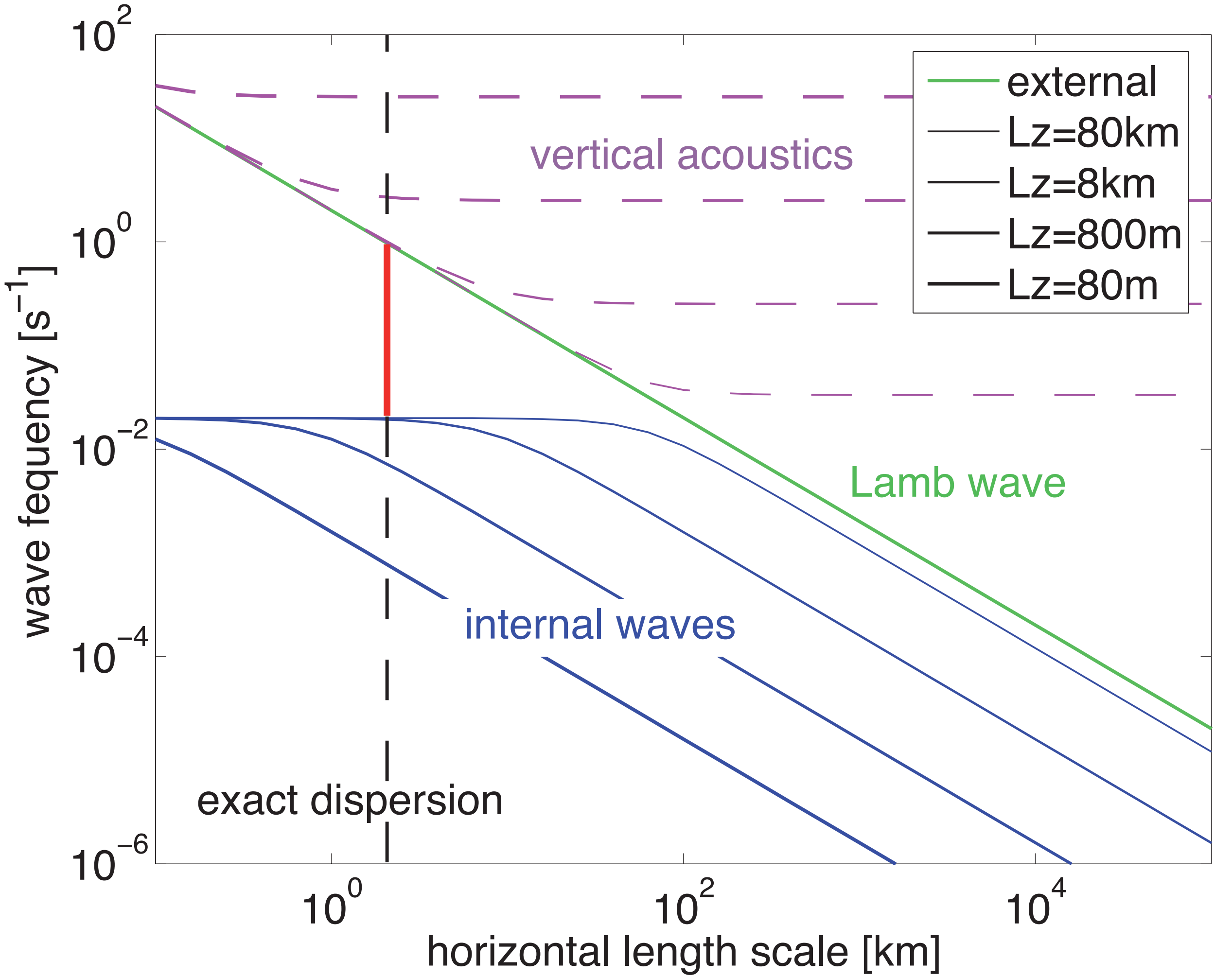
Summary

Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



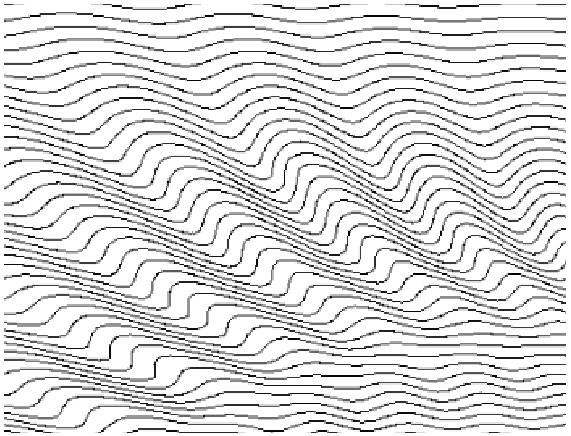
Time scale gap for short wave lengths $L \sim 2\pi$ km



Wave breaking regime, strong stratification

WKB theory:

- $\sim 2\pi$ km wave packets
- modulated over ~ 10 km distances
- ***θ*–stratification of order $O(1)$**
- scalings allow for overturning of θ-contours



Expansion scheme:

$$U(t, \boldsymbol{x}, z; \varepsilon) = \overline{U}(z) + U_1^{(0)} \exp\left(i\frac{\varphi^\varepsilon}{\varepsilon}\right) + \varepsilon \sum_{n=0}^{2} U_n^{(1)} \exp\left(in\frac{\varphi^\varepsilon}{\varepsilon}\right)$$

$$\varphi^\varepsilon = \varphi^{(0)} + \varepsilon\varphi^{(1)} + o(\varepsilon)$$

$$\left(U_n^{(i)}, \varphi^{(i)}\right) \equiv \left(U_n^{(i)}, \varphi^{(i)}\right)(t, \boldsymbol{x}, z)$$

Wave breaking regime, strong stratification

Leading order: — classical Boussinesq / ray tracing theory

$$\underbrace{\begin{pmatrix} -i\hat{\omega} & 0 & 0 & ik \\ 0 & -i\hat{\omega} & -N & im \\ 0 & N & -i\hat{\omega} & 0 \\ ik & im & 0 & 0 \end{pmatrix}}_{M(\hat{\omega}, k, m)} \begin{pmatrix} \hat{U}^{(0)} \\ \hat{W}^{(0)} \\ \frac{1}{N}\frac{\hat{\Theta}^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}^{(2)} \end{pmatrix} = 0 \qquad \text{where} \qquad \begin{cases} \hat{\omega} = -\dfrac{\partial \varphi^{(0)}}{\partial t} - k u_0^{(0)} \\ k = \dfrac{\partial \varphi^{(0)}}{\partial \boldsymbol{x}} \\ m = \dfrac{\partial \varphi^{(0)}}{\partial z} \end{cases}$$

Wave breaking regime, strong stratification

First order:

$$M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}$$

Wave breaking regime, strong stratification

First order: **phase** corrections

$$M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + \underline{M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)}} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}$$

- First-order Hamilton-Jacobi-eqn. for $\varphi^{(1)}$

Wave breaking regime, strong stratification

First order: **pseudo-incompressible** corrections

$$M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)} \frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)} \frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)} \frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)} \frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)} \frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)} \frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ \boldsymbol{-\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta}} \end{pmatrix}$$

- **pseudo-incompressible** wave action conservation law

Wave breaking regime, strong stratification

First order: **higher harmonics**

$$M^{(0)}\begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)}\begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}$$

$+$ **Nonlinear Effects:**

Explicit solutions for all higher-oder modes $\sim \exp\left(i\, n\, \varphi^{(1)}/\varepsilon\right)$, $\qquad (n = 1, 2, \ldots)$

Regime(s) of validity of sound-proof models

Freie Universität Berlin

Compressible flow equations

$$
\begin{aligned}
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\end{aligned}
$$

drop term for:

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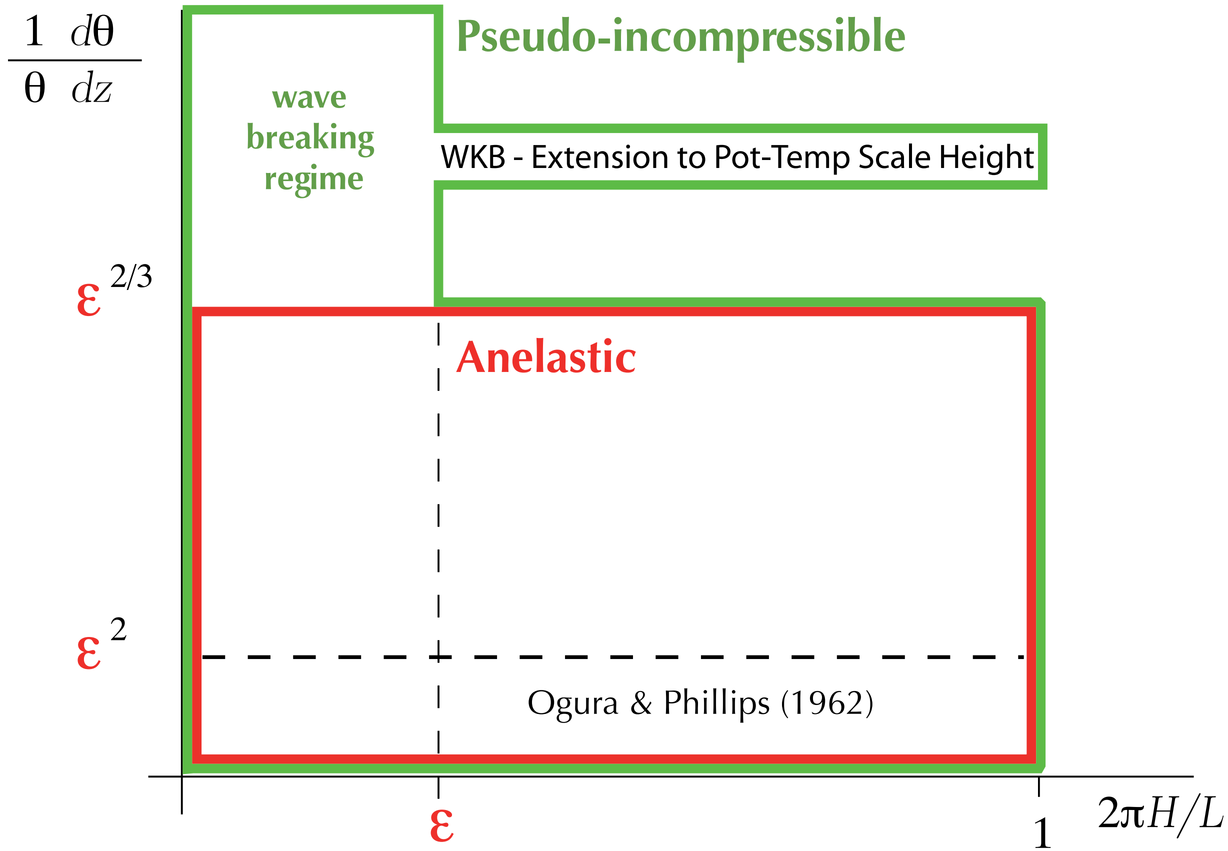
pseudo-incompressible

$$
P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k}\,, \quad (\boldsymbol{u}\cdot\boldsymbol{k}\equiv 0)
$$

Summary



Remarks:

- Isothermal stratification captured for

$$\frac{\gamma - 1}{\gamma} = O\left(\varepsilon^{2/3}\right)$$

- Uniform approximation in the long wave limit

Pseudo-incompressible model wins by small margin